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The Inverse Problem of Scattering Theory

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DEDICATION

This book is dedicated to the illustrious memory of the Rector of the Kharkov State University, Ivan Nikolayevich Bulankin.

Z.S.A. and V.A.M.

Preface

In spectral theory, the inverse problem is the usual name for any problem in which it is required to ascertain the spectral data that will determine a differential operator uniquely and a method of constructing this oparator from the data.

A problem of this kind was first formulated and investigated by V. A. Ambartsumian in 1929. Since 1946, various forms of the inverse problem have been considered by numerous foreign authors (G. Borg, V. Bargmann, N. Levinson, etc.) and Soviet authors (V. A. Marchenko, M. G. Krein, I. M. Gelfand, B. M. Levitan, etc.), and there now exists an extensive literature on the question.

No attempt is made in this monograph to review the work done on the inverse problem. Instead, merely one of its variants will be treated and solved, namely, the problem arising in connection with the quantum theory of scattering and which is apparently the most interesting from the standpoint of application. The mathematical techniques developed in the solution of the problem may also be applied to related questions.

Translator's preface

The basic question treated in this book, a translation of the monograph entitled *Obratnaya zadacha teorii rasseyaniya,* is encountered in many fields. Besides the quantum theoretical problem, there is, for example, the electromagnetic inverse scattering problem, i. e., the problem of determining information about a medium from which an electromagnetic wave is reflected, given a knowledge of the reflection coefficient. The authors Agranovich and Marchenko have presented a comprehensive lucid solution of another such problem arising in the theory of the deuteron. It is based mainly on the considerable amount of work they have done in this and related areas. Moreover, the functional analytic and algebraic methods used should also be of great interet to pure and applied mathematicians.

The content of the translation remains the same as in the Russian original except for some stylistic changes and minor additions for purposes of clarity. However, an attempt was made to find and eliminate all typographical errors. Also, a subject index and notation index were added, the latter covering the symbols used most frequently throughout the book.

B. D. Seckler

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Introduction

The inverse problem of scattering theory[1] originates in theoretical physics.

It is well known in quantum mechanics that the scattering of particles by a potential field is completely determined by the asymptotic form of the wave functions at infinity. In accordance with Heisenberg's ideas (1943—1946), it is precisely the asymptotic behavior of the wave functions that has physical meaning.

The question therefore naturally arises as to whether it is possible to reconstruct the potential from a knowledge of the asymptotic form of the wave functions at infinity, and if this is possible, to give a method for carrying out the construction. Such is the inverse problem of scattering theory. Of even more interest than the actual construction of the potential are the various relationships that can be established between it and the asymptotic form of the wave functions.

Aside from these remarks, we shall make no further mention of the physical aspects of the problem. We shall be interested only in the mathematical side of the question which can be formulated as follows.

Consider the boundary-value problem described by the system of differential equations

$$y_j'' + \lambda^2 y_j = \sum_{k=1}^{n} v_{jk}(x) y_k, \qquad 0 < x < \infty \; (j = 1, 2, \ldots n), \tag{1}$$

---

[1] For a survey and short history of the problem, see L.D. Faddeyev [15]. (Numbers in brackets refer to the bibliography at the end of the book).

with a Hermitian coefficient matrix (potential matrix)

$$V(x) = \| v_{jk}(x) \|_1^n,$$

and boundary condition

$$y_j(0) = 0 \qquad (j = 1,\ 2, \ldots n). \tag{2}$$

Knowing the asymptotic behavior at infinity of the normalized eigenfunctions of the boundary-value problem (1), (2), we are required to determine the potential matrix $V(x)$.

In the case most interesting from the physical standpoint, $V(x)$ has singularities of the order x^{-2} in the vicinity of zero and infinity. The boundary-value problem with these particular singularities will be considered in the second part of the book (Chapters VII and VIII). However, in the first part (Chapters I—V), we shall treat the mathematically simpler case in which the condition

$$\int_0^\infty x \mid v_{jk}(x) \mid dx < \infty \qquad (j,\ k = 1,\ 2, \ldots n) \tag{3}$$

is satisfied. The results obtained there not only are of independent interest, but also serve as a basis for the investigation of the singular boundary-value problem.

In Chapters II and IV it is shown that when condition (3) is satisfied, the boundary-value problem (1), (2) has a finite number of negative eigenvalues λ_k^2, with Im $\lambda_k < 0$, and a continuous spectrum coinciding with the entire positive axis. The normalized eigenfunctions of the problem are the columns of matrices $U(x,\lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1,2 \ldots\ldots p$) for which Parseval's equality holds in a form equivalent to the following expansion of the δ-function:

$$\delta(x-y)I = \sum_{k=1}^{p} U(x,\ \lambda_k)U^*(y,\ \lambda_k) + \frac{1}{2\pi}\int_0^\infty U(x,\ \lambda)U^*(y,\ \lambda)\, d\lambda \tag{4}$$

(here I is the identity matrix). Moreover, as $x \to \infty$

$$\left.\begin{aligned} U(x,\ \lambda) &= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o\,(1) \qquad (\lambda > 0), \\ U(x,\ \lambda_k) &= e^{-|\lambda_k|x}[M_k + o\,(1)] \qquad (k = 1,\ 2, \ldots p), \end{aligned}\right\} \tag{5}$$

where $S(\lambda)$ is a unitary matrix for $-\infty < \lambda < \infty$ with the property that $S(-\lambda) = S^*(\lambda)$, and M_k is a non-negative Hermitian matrix of rank equal to the multiplicity of the eigenvalue λ_k^2. $S(\lambda)$ is called the *scattering matrix*, M_k are the *normalization matrices*, and the aggregate of quantities $S(\lambda)$, λ_k^2, and M_k ($k = 1,2 \ldots\ldots p$) constitutes the *scattering data* of the boundary-value problem (1), (2).

Thus, for the case in question, the inverse scattering problem becomes the following:

1. Determine a method of constructing the potential $V(x)$ from the scattering data of the boundary-value problem (1), (2).

2. Find necessary and sufficient conditions for a given unitary matrix $S(\lambda)$, negative numbers λ_k^2, and Hermitian matrices M_k ($k = 1,2, \ldots, p$) to be the scattering data of some problem (1), (2), in which the Hermitian matrix $V(x)$ satisfies condition (3).

The first of these problems was considered in a paper of Newton and Jost [1]. Imposing stricter conditions on the potential than (3), these authors reduced the problem to that of determining the spectral matrix from the scattering data from which the potential could then be obtained by the method of Gelfand and Levitan [2]. Some special cases of this problem were treated by others using the methods of Krein [3,4].

The method to be used in this book[2] permits the construction of the potential directly from the scattering data and yields the complete solution to the above second problem of determining the characteristic properties of the scattering data.

The starting point is the following theorem,[3] which will be proved in § 3, Chapter I.

If the condition (3) holds, then a solution $E(x, \lambda)$ of the matrix differential equation

$$Y'' + \lambda^2 Y = V(x)Y, \qquad 0 < x < \infty \tag{6}$$

exists which can be represented as

$$E(x, \lambda) = e^{-i\lambda x}I + \int_x^\infty K(x, t)e^{-i\lambda t}\,dt \qquad (\operatorname{Im}\lambda \leq 0), \tag{7}$$

---

[2] The idea behind this method is presented for a single equation ($n = 1$) in the article [5]. The basic results obtained by this method for a system of equations appear in [6].

[3] This theorem for the scalar equation $y'' + \lambda^2 y = v(x)y$ was first proved by Levin [7] under the assumption that

$$\int_0^\infty (1 + x^2)\,|v(x)|\,dx < \infty.$$

the matrix $K(x, t)$ and potential $V(x)$ being related by

$$V(x) = -2\frac{d}{dx}K(x, x) \qquad (x > 0). \tag{8}$$

The procedure set forth for constructing the potential is based on the fact that $K(x, t)$ satisfies a linear integral equation (the *fundamental equation*)

$$F(x+y)+K(x, y)+\int_x^\infty K(x, t)F(t+y)\,dt = 0 \qquad (0 < x \leq y), \tag{9}$$

in which $F(t)$ is determined by the scattering data according to the formula

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k| t}+\frac{1}{2\pi}\int_{-\infty}^{\infty}[I-S(\lambda)]e^{i\lambda t}\,d\lambda. \tag{10}$$

The derivation and investigation of (9) is carried out in Chapter III.

It is clear that the fundamental equation (9) and the formula (8) together yield the solution to the first of the above problems, i.e., they enable $V(x)$ to be constructed from the scattering data, and moreover, in a very simple way.

The analysis carried out in Chapters II—IV shows that the scattering data for the boundary-value problem (1), (2) [in which the Hermitian potential $V(x)$ satisfies the condition (3)] has the five following properties:

I_s. The matrix $I-S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$; each element of the matrix

$$F_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[I-S(\lambda)]e^{i\lambda t}\,d\lambda \tag{11}$$

is summable over the interval $(0, \infty)$ and can be expressed in the interval $(-\infty, 0)$ as the sum of two functions, one of which is summable and the other square summable and bounded.

II_s. The derivative $F_s'(t)$ exists for all $t > 0$ and [4]

$$\int_0^\infty t\,|F'(t)|\,dt < \infty.$$

---

[4] We shall define the absolute value of a matrix $A = \|a_{jk}\|$ by the formula $|A| = \max_j \sum_k |a_{jk}|$.

III_s. The equation

$$-x(t)+\int_{-\infty}^{0} x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad -\infty < t < 0$$

has no non-trivial vector solutions with components that are square summable over the interval $(-\infty, 0)$.

IV. The equation

$$x(t)+\int_{0}^{\infty} x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 < t < \infty$$

has no non-trivial vector solutions with components summable over the interval $(0, \infty)$.

V. The number of linearly independent solutions of

$$x(t)+\int_{0}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad 0 < t < \infty$$

with components summable over the interval $(0, \infty)$, is equal to the sum of the ranks of the normalization matrices $M_1, M_2, \ldots, M_p$

In Chapter V, it is shown that these five conditions are not only necessary but also sufficient for a given unitary matirx $S(\lambda)$, negative numbers λ_k^2, and Hermitian matrices M_k $(k = 1,2, \ldots, p)$ to be the scattering data for some boundary-value problem (1), (2) with a Hermitian potential $V(x)$ satisfying condition (3). At that time, the role of each of the above five properties will be clarified.

The results of Chapter III already show that if the unitary matrix $S(\lambda)$ satisfies merely condition I_s and M_k and $\lambda_k^2 < 0$ are prescribed arbitrarily $(k = 1,2, \ldots, p)$, then the fundamental equation (9) [with $F(t)$ defined in terms of the above data by means of (10)] has a unique solution $K(x, y)$ for each positive value of x. Once $K(x, y)$ is known, it is possible to obtain $E(x, y)$ from formula (7) and then the matrices

$$\begin{aligned} U(x, \lambda) &= E(x, -\lambda)-E(x, \lambda)S(-\lambda) \qquad (\lambda > 0),\\ U(x, \lambda_k) &= E(x, \lambda_k)M_k \qquad (k = 1, 2, \ldots p), \end{aligned} \tag{12}$$

(Chapter IV, § 2), which clearly satisfy the asymptotic relations (5) and yield Parseval's equality (4).

If the matrix $S(\lambda)$ has the properties I_s and II_s, then the matrices $U(x, \lambda)$ defined by (12) are solutions of equation (6) with a Hermitian potential $V(x) = -2\, dK(x, x)/dx$ satisfying the condition

$$\int_{\varepsilon}^{\infty} x\,|V(x)|\,dx < \infty$$

for any positive ε. If, in addition, the given data possesses property IV, $V(x)$ will also satisfy this inequality with $\varepsilon = 0$, i.e., it will satisfy condition (3). [We note that if the condition IV does not hold, then $V(x)$ will not satisfy (3)].

In general, the conditions I_s, II_s, and IV are not sufficient for $U(x, \lambda)$ and $U(x, \lambda_k)$ to satisfy the boundary condition (2). The fulfillment of the boundary condition on the discrete spectrum, i.e., the equality

$$U(0, \lambda_k) = 0 \qquad (k = 1, 2, \ldots p),$$

is assured if the given data has the properties I_s, IV. and V; if, moreover, property III_s holds, then the boundary condition is also fulfilled on the continuous spectrum, i.e.,

$$U(0, \lambda) = 0 \qquad (0 < \lambda < \infty).$$

In § 6, Chapter V, it is further shown that I_s, II_s and III_s are necessary and sufficient for a unitary matrix $S(\lambda)$ to be the scattering matrix for some boundary-value problem (1), (2) with a Hermitian potential satisfying condition (3). In other words, if $S(\lambda)$ is a given unitary matrix having the three stated properties, then it is always possible to find negative numbers $\lambda_1^2, \lambda_2^2, \ldots, \lambda_p^2$ and Hermitian matrices $M_1, M_2, \ldots, M_p$ which together with $S(\lambda)$ have the properties I_s, II_s, III_s, IV and V.

A more detailed analysis in § 6, Chapter V shows also that if $S(\lambda)$ has the properties I_s, II_s, and III_s, then the matrix $F_s(t)$ determined by (11) is in fact summable over the entire real line and $S(\lambda)$ is therefore uniformly continuous.

Concerning the actual solution of the fundamental equation (9), we note the following: If the elements of $S(\lambda)$ are rational functions, then the kernel of the fundamental equation $F(t+y)$ is degenerate, and the equation can be solved by elementary means. It is also obvious that the fundamental equation can always be solved for large x by the method of successive approximations. It is further shown (Theorem 3.4.2) that if no discrete spectrum exists, i.e., $M_k = 0$ for all k, then the fundamental equation can be solved by the method of successive approximations for any positive value of x.

In the second part of the book, a method of investigating boundary problems with singularities is developed (i.e., systems of equations of the form (1) in which the coefficients have singularities of the order x^{-2} in the neighborhood of zero and infinity).

The procedure consists in applying transformations of a special type which convert the boundary-value problem in question into one without singularities (such as is investigated in the first part of the book) while enabling one to observe how the spectrum and asymptotic form of the normalized eigenfunctions change as the singular problem is transformed. General information concerning these transformations is contained in Chapter VI.

In the following chapters, the above method is applied to a boundary-value problem consisting of a system of two equations which in matrix form is

$$Y''+\lambda^2 Y = [V(x)+6x^{-2}P]Y, \qquad 0 < x < \infty. \tag{13}$$

Here $V(x) = \| v_{jk}(x) \|_1^2$ is a Hermitian matrix satisfying the condition

$$\int_0^\infty x^{1+\theta} \,|\, V(x) \,|\, dx < \infty \qquad (-\varepsilon < \theta < \varepsilon), \tag{14}$$

for some value of ε with $0 < \varepsilon < 1$ and

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is well known [8,9] that the Schrioedinger equation for a deuteron (in its ground state) leads to such a system if tensor interaction forces are taken into account.

Chapters VII and VIII will be concerned with obtaining a few concrete results pertinent to the system (13) [5] and primarily with a demonstration of how the above method works. This method may also be applied to other analogous boundary-value problems with singularities.

In Chapter VII, the spectral analysis of the boundary-value problem consisting of the system [13] and boundary condition

$$Y(0) = 0 \tag{15}$$

is carried out by this method, and the following is established.

---

[5] These results were published without proof in [10].

The boundary-value problem (13), (15) [in which $V(x)$ satisfies the condition (14)] has a continuous spectrum coinciding with the entire positive real axis ($\lambda^2 > 0$) and, possibly, a finite number of non-positive eigenvalues $\lambda_1^2 < \lambda_2^2 < \ldots < \lambda_p^2 \leq 0$. With λ^2 in the the spectrum, there exist matrix solutions $U(x, \lambda)$ of equation (13) which vanish for $x = 0$, for which Parseval's equality (4) holds, and which satisfy the relations

$$\begin{aligned} U(x, \lambda) &= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) && (\lambda^2 > 0), \\ U(x, \lambda_k) &= e^{-|\lambda_k|x}[M_k + o(1)]. && (\lambda_k^2 < 0) \end{aligned} \qquad (x \to \infty).$$

If $\lambda_p = 0$, then

$$U(x, \lambda_p) = U(x, 0) = x^{-2}[M_p + o(1)] \qquad (M_p = mP,\ m > 0).$$

Here, $S(\lambda)$ is a unitary matrix for $-\infty < \lambda < \infty$ with the property that $S(-\lambda) = S^*(\lambda)$, and M_k is a non-negative Hermitian matrix of rank equal to the multiplicity of the eigenvalue $\lambda_k^2 \cdot S(\lambda)$ is the scattering matrix, M_k are the normalization matrices, and the aggregate of quantities $S(\lambda)$, λ_k^2, and M_k ($k = 1,2, \ldots, p$) comprises the scattering data for the problem (13), (15).

Along with the spectral analysis of the boundary-value problem (13), (15), a study is made of the properties of its scattering data, and it is shown that these properties split up into three different possible cases. These results are formulated in sub-section 2 of § 7, Chapter VII.

In the last chapter, the inverse problem for (13), (15) is solved. It is shown there that the conditions obtained in the preceding chapter not only are necessary but also sufficient for a given unitary matrix $S(\lambda)$, non-positive numbers λ_k^2, and Hermitian matrices M_k ($k = 1,2, \ldots, p$) to be the scattering data for some boundary-value problem (13), (15) [with a Hermitian matrix $V(x)$ satisfying condition (14)]. At the same time, an alogrithm is presented for determining $V(x)$ from prescribed scattering data.

We note that the scattering data is not always uniquely determined. In certain cases, there exists a one-parameter family of Hermitian matrices $V(x)$ satisfying the condition (14) such that the corresponding boundary-value problems yield identical scattering data.

There are two appendices to the book. The first is devoted to a modification of the set of characteristic properties of the scattering data of the problem without singularities. In this connection, several results are also obtained on the factorization of a unitary matrix. In the second appendix, some important inequalities, used in the first part of the book to investigate the problem without singularities, are sharpened.

To simplify the presentation, we shall assume that the potential $V(x)$ is continuous in the interval $0 < x < \infty$. It is clear that this restriction may easily be removed.

PART ONE

The Boundary-Value Problem without Singularities

CHAPTER I

Particular Solutions of the System without Singularities

§ 1. Preliminary Remarks and Notation

Consider the system of differential equations

$$y_j'' + \lambda^2 y_j = \sum_{k=1}^{n} v_{jk}(x) y_k \qquad (j = 1, 2, \ldots n), \tag{1.1.1}$$

where the matrix of functions

$$V(x) = \| v_{jk}(x) \|_1^n$$

is defined and continuous[1] in the interval $0 < x < \infty$, and λ is an arbitrary parameter. In the following, we shall call $V(x)$ the *potential matrix* or, simply, the *potential.*

Every set of n solutions of the system can be represented as a square matrix of n-th order $Y(x, \lambda)$ satisfying the equation

$$Y'' + \lambda^2 Y = V(x) Y, \qquad 0 < x < \infty. \tag{1.1.2}$$

Conversely, the columns of every matrix solution of equation (1.1.2) are clearly solutions of the system (1.1.1).

Thus we can study the matrix differential equation (1.1.2) instead of the system (1.1.1), and this proves to be more convenient.

The *absolute value* of the matrix $A = \| a_{jk} \|$ (which, generally speaking, is rectangular) will be defined as the non-negative number

$$\max_j \sum_k | a_{jk} |$$

---

[1] A matrix is said to be continuous if all its elements are continuous functions. In the same sense, we shall refer to a matrix as being summable, differentiable, regular, etc.

which we shall denote by $|A|$. The determinant of A will be denoted by $\det A$.

It is easy to show that

$$|A+B| \leq |A| + |B|$$
$$|A\cdot B| \leq |A| \cdot |B|$$

(provided, of course, the expressions $A+B$ and $A\cdot B$ are meaningful), and that if the matrix $A(x)$ is summable over the interval (a, b), then

$$\left|\int_a^b A(x)\,dx\right| \leq \int_a^b |A(x)|\,dx.$$

In the investigation of the equation (1.1.2), we shall make systematic use of the notation

$$\sigma(x) = \int_x^\infty |V(t)|\,dt; \qquad \sigma_1(x) = \int_x^\infty t\,|V(t)|\,dt. \tag{1.1.3}$$

Of other notation to be used below, we mention the following: I is the identity matrix, A^* is the complex conjugate transpose of the matrix A, and $W\{Y, Z\}$ is the Wronskian of the differentiable matrices $Y(x)$ and $Z(x)$, i.e.,

$$W\{Y, Z\} = Y(x)Z'(x) - Y'(x)Z(x). \tag{1.1.4}$$

The inequality $\int^\infty |f(t)|\,dt < \infty$ will mean that $f(t)$ is summable over the interval (a, ∞), and $\int_0 |f(t)|\,dt < \infty$ that it is summable over the interval $(0, a)$ for any positive value of a.

We shall use the symbol $o\{\varphi(x)\}$ in matrix equations to denote the fact that all the elements of a matrix are $o\{\varphi(x)\}$.

§ 2. A Fundamental System of Solutions with a Prescribed Behavior in the Vicinity of Zero

THEOREM 1.2.1: *If for some* $\varepsilon \geq 0$ $(\varepsilon < 1)$ [2]

$$\int_0 x^{1-\varepsilon}\,|V(x)|\,dx < \infty, \tag{1.2.1}$$

then the equation (1.1.2) *has a fundamental system of solutions* $G(x, \lambda)$

---

[2] The theorem is true for $0 \leq \varepsilon < 2$ but we shall not need this fact later on.

and $H(x, \lambda)$ *for each (fixed) value of* λ, *satisfying the asymptotic relations*

$$\left.\begin{aligned} G(x, \lambda) &= x[I+o(x^{\epsilon})], & G'(x, \lambda) &= I+o(x^{\epsilon}); \\ H(x, \lambda) &= I+o(x^{\epsilon}), & H'(x, \lambda) &= o(x^{-1+\epsilon}). \end{aligned}\right. \quad (x \to 0)$$

For each fixed value of $x \geq 0$, $G(x, \lambda)$ *is an entire function of* λ.

Proof: Consider the integral equation

$$G(x, \lambda) = \frac{\sin \lambda x}{\lambda} I + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} V(t)G(t, \lambda)\, dt, \qquad (1.2.2)$$

which is easily seen to be equivalent to the differential equation (1.1.2) with the initial conditions

$$G(0, \lambda) = 0, \qquad G'(0, \lambda) = I.$$

We first consider the case Im $\lambda \leq 0$, and we let

$$G(x, \lambda) = xe^{i\lambda x}Z(x, \lambda).$$

Equation (1.2.2) then yields

$$Z(x, \lambda) = \frac{\sin \lambda x}{\lambda x} e^{-i\lambda x}I +$$

$$+ \int_0^x \frac{\sin \lambda(x-t)}{\lambda x} e^{-i\lambda(x-t)}t V(t)Z(t, \lambda)\, dt. \qquad (1.2.3)$$

We now seek a solution of this equation in series form

$$Z(x, \lambda) = \sum_{k=0}^{\infty} Z_k(x, \lambda), \qquad (1.2.4)$$

using the method of successive approximations. Here, by definition,

$$Z_0(x, \lambda) = \frac{\sin \lambda x}{\lambda x} e^{-i\lambda x}I,$$

$$Z_k(x, \lambda) = \int_0^x \frac{\sin \lambda(x-t)}{\lambda x} e^{-i\lambda(x-t)}t V(t)Z_{k-1}(t, \lambda)\, dt.$$

Since for Im $\lambda \leq 0$ and $0 \leq t \leq x$, we have

$$\left|\frac{\sin \lambda x}{\lambda x} e^{-i\lambda x}\right| = \left|\frac{1}{x}\int\limits_0^x e^{-2i\lambda s}\, ds\right| \leq 1$$

$$\left|\frac{\sin \lambda(x-t)}{\lambda x} e^{-i\lambda(x-t)}\right| \leq 1-\frac{t}{x} \leq 1,$$

it follows that the series (1.2.4) is majorized by the series

$$\sum_{k=0}^{\infty} \zeta_k(x),$$

where

$$\zeta_0(x) = |I| = 1, \quad \zeta_k(x) = \int\limits_0^x t\,|V(t)|\,\zeta_{k-1}(t)\, dt. \qquad (k = 1, 2, \ldots)$$

This last series converges and, moreover, uniformly in every finite interval $0 \leq x \leq a$ since

$$0 \leq \zeta_k(x) \leq \frac{1}{k!}\left[\int\limits_0^x t\,|V(t)|\, dt\right]^k.$$

Hence, the series (1.2.4) converges uniformly for $0 \leq x \leq a$, Im $\lambda \leq 0$ and its sum $Z(x,\lambda)$ is a solution of (1.2.3). It is also evident that $Z(x, \lambda)$ satisfies the inequality

$$|Z(x, \lambda)| \leq e^{\int\limits_0^x t\,|V(t)|\,dt}.$$

But this means that $G(x,\lambda) = xe^{i\lambda x} Z(x,\lambda)$ is a solution of (1.2.2) and satisfies the inequality

$$|e^{-i\lambda x}G(x, \lambda)| \leq x e^{\int\limits_0^x t\,|V(t)|\,dt} \qquad (x \geq 0,\ \mathrm{Im}\,\lambda \leq 0). \qquad (1.2.5)$$

From the uniform convergence of the series (1.2.4), it further follows that its sum $Z(x, \lambda)$ (and therefore $G(x, \lambda)$ as well) is regular in the half-plane Im $\lambda \leq 0$ for each fixed value of $x \geq 0$ and is continuous there up to the real axis, since every term of the series has these properties.

In an analogous way, it can be shown that a solution to (1.2.2) exists for Im $\lambda \geq 0$, and that its solution $G(x, \lambda)$ for each fixed value of $x \geq 0$ is regular in the half-plane Im $\lambda \geq 0$ and is continuous there up to the real axis. Hence, $G(x,\lambda)$ is an entire function of λ for each fixed value of $x \geq 0$.

The required asymptotic behavior of $G(x, \lambda)$ as $x \to 0$ is an immediate consequence of (1.2.2), the equation

$$G'(x, \lambda) = \cos \lambda x I + \int_0^x \cos \lambda(x-t) V(t) G(t, \lambda)\, dt,$$

the condition (1.2.1), and the inequality (1.2.5).

The matrix $H(x,\lambda)$ is defined as the solution of the integral equation

$$H(x, \lambda) = \cos \lambda x I - \cos \lambda x \int_0^x \frac{\sin \lambda t}{\lambda} V(t) H(t, \lambda)\, dt - \\ - \frac{\sin \lambda x}{\lambda} \int_x^h \cos \lambda t V(t) H(t, \lambda)\, dt, \tag{1.2.6}$$

any solution of which is also a solution of (1.1.2). Let us show that it is possible to solve (1.2.6) by the method of successive approximations for sufficiently small values of x lying in some interval $0 \leq x \leq h$ whenever

$$e^{2h|\lambda|} \int_0^h t\, |V(t)|\, dt = q < 1.$$

In fact, if we define

$$H_0(x, \lambda) = \cos \lambda x I,$$

$$H_k(x, \lambda) = -\cos \lambda x \int_0^x \frac{\sin \lambda t}{\lambda} V(t) H_{k-1}(t, \lambda)\, dt - \\ - \frac{\sin \lambda x}{\lambda} \int_x^h \cos \lambda t V(t) H_{k-1}(t, \lambda)\, dt \qquad (k = 1, 2, \ldots),$$

we find that for $0 \leq x \leq h$

$$|H_0(x, \lambda)| = |\cos \lambda x I| \leq e^{x|\lambda|} \leq e^{h|\lambda|},$$

$$|H_1(x, \lambda)| \leq e^{x|\lambda|} \int_0^x t e^{t|\lambda|} |V(t)|\, e^{t|\lambda|}\, dt + x e^{x|\lambda|} \int_x^h e^{t|\lambda|} |V(t)|\, e^{t|\lambda|}\, dt \leq \\ \leq e^{3h|\lambda|} \int_0^h t\, |V(t)|\, dt = e^{h|\lambda|} q.$$

By mathematical induction, we then obtain

$$|H_k(x, \lambda)| \leq e^{h|\lambda|} q^k \qquad (k = 0, 1, 2, \ldots).$$

The above estimates imply that the series

$$\sum_{k=0}^{\infty} H_k(x, \lambda)$$

converges for fixed λ uniformly in the interval $0 \leq x \leq h$, and its sum $H(x, \lambda)$ is therefore a solution of (1.2.6) and hence of the equation (1.1.2). If is further obvious that for fixed λ, $H(x, \lambda)$ is bounded in the interval $0 \leq x \leq h$.

Now consider the behavior of $H(x, \lambda)$ in the neighborhood of $x = 0$. From (1.2.6), we have

$$\begin{aligned} |H(x, \lambda) - I| &\leq |\cos \lambda x - 1| + e^{x|\lambda|} \int_0^x t e^{t|\lambda|} |V(t)| |H(t, \lambda)| \, dt + \\ &\quad + x e^{x|\lambda|} \int_x^h e^{t|\lambda|} |V(t)| |H(t, \lambda)| \, dt \leq \\ &\leq |\cos \lambda x - 1| + x^{\varepsilon} e^{x|\lambda|} \int_0^x t^{1-\varepsilon} e^{t|\lambda|} |V(t)| |H(t, \lambda)| \, dt + \\ &\quad + x^{\varepsilon} e^{x|\lambda|} \left\{ \int_x^{\sqrt{x}} t^{1-\varepsilon} e^{t|\lambda|} |V(t)| |H(t, \lambda)| \, dt + \right. \\ &\quad \left. + x^{\frac{1+\varepsilon}{2}} \int_{\sqrt{x}}^h t^{1-\varepsilon} e^{t|\lambda|} |V(t)| |H(t, \lambda)| \, dt \right\}. \end{aligned}$$

Hence, by (1.2.1), it follows that for $\varepsilon < 1$

$$H(x, \lambda) = I + o(x^{\varepsilon}) \qquad (x \to 0).$$

In a similar way, we deduce from the relation

$$\begin{aligned} H'(x, \lambda) = -\lambda \sin \lambda x I + \sin \lambda x \int_0^x \sin \lambda t V(t) H(t, \lambda) \, dt - \\ - \cos \lambda x \int_x^h \cos \lambda t V(t) H(t, \lambda) \, dt \end{aligned}$$

that

$$H'(x, \lambda) = o(x^{-1+\varepsilon}) \qquad (x \to 0).$$

This completes the proof of the theorem.

The following lemma will also be needed later.

LEMMA 1.2.1: *For all* $x \geq 0$ *and* $\operatorname{Im}\lambda \leq 0$,

$$| \lambda e^{-i\lambda x} G(x, \lambda) | \leq e^{\int\limits_0^x t\,|V(t)|\,dt} . \tag{1.2.7}$$

Furthermore, for each fixed value of $x \geq 0$,

$$| \lambda e^{-i\lambda x} G(x, \lambda) - \sin \lambda x \, e^{-i\lambda x} I | \to 0 \tag{1.2.8}$$

as $|\lambda| \to \infty$, *uniformly in the region* $\operatorname{Im}\lambda \leq 0$.

Proof: The integral equation (1.2.2) can be expressed as

$$\lambda e^{-i\lambda x} G(x, \lambda) = \sin \lambda x \, e^{-i\lambda x} I + $$

$$+ \int\limits_0^x \sin \lambda(x-t) e^{-i\lambda(x-t)} V(t) \, [e^{-i\lambda t} G(t, \lambda)] \, dt, \tag{1.2.9}$$

and hence, by virtue of (1.2.5) and the inequality

$$| \sin \lambda x \cdot e^{-i\lambda x} | \leq 1 \qquad (\operatorname{Im}\lambda \leq 0),$$

we obtain

$$| \lambda e^{-i\lambda x} G(x, \lambda) | \leq | I | + \int\limits_0^x t \, | V(t) | \, e^{\int\limits_0^t s\,|V(s)|\,ds} dt =$$

$$= 1 + e^{\int\limits_0^x s\,|V(s)|\,ds} - 1 = e^{\int\limits_0^x t\,|V(t)|\,dt} ,$$

i.e., the inequality (1.2.7).

The relation (1.2.8) is trivially true for $x = 0$. However, if $x > 0$, then taking $\operatorname{Im}\lambda \leq 0$, $\sqrt{|\lambda|} > 1/x$ and making use of the estimates (1.2.5) and (1.2.7), we find from (1.2.9) that

$$| \lambda e^{-i\lambda x} G(x, \lambda) - \sin \lambda x \, e^{-i\lambda x} I | \leq$$

$$\leq \int\limits_0^{\frac{1}{\sqrt{|\lambda|}}} | V(t) | \, | e^{-i\lambda t} G(t, \lambda) | \, dt + \frac{1}{|\lambda|} \int\limits_{\frac{1}{\sqrt{|\lambda|}}}^x | V(t) | \, | \lambda e^{-i\lambda t} G(t, \lambda) | \, dt \leq$$

$$\leq e^{\int\limits_0^x t\,|V(t)|\,dt} \left\{ \int\limits_0^{\frac{1}{\sqrt{|\lambda|}}} t \, | V(t) | \, dt + \frac{1}{\sqrt{|\lambda|}} \int\limits_{\frac{1}{\sqrt{|\lambda|}}}^x t \, | V(t) | \, dt \right\}.$$

As $|\lambda| \to \infty$, the right-hand side of this inequality clearly tend to zero uniformly in the half-plane $\operatorname{Im}\lambda \leq 0$.

§ 3. The Special Solution and Transformation Operator

Theorem 1.3.1: *If*

$$\sigma_1(x) = \int_x^\infty t\,|V(t)|\,dt < \infty \qquad (x > 0), \tag{1.3.1}$$

then the equation (1.1.2) *has a solution* $E(x, \lambda)$ *given by*

$$E(x, \lambda) = e^{-i\lambda x}I + \int_x^\infty K(x, t)e^{-i\lambda t}\,dt \qquad (\operatorname{Im}\lambda \leq 0), \tag{1.3.2}$$

where the matrix $K(x, t)$ *satisfies the inequality*

$$|K(x, t)| \leq \frac{1}{2}e^{\sigma_1(x)}\sigma\left(\frac{x+t}{2}\right) \quad (\text{see } 1.1.3). \tag{1.3.3}$$

For each positive value of x *(and for* $x = 0$*, as well, in the event that* $\sigma_1(0) < \infty$*),* $E(x, \lambda)$ *is regular in the half-plane* $\operatorname{Im}\lambda < 0$ *and continuous there up to and including the real axis.*

Proof: Consider the integral equation

$$E(x, \lambda) = e^{-i\lambda x}I + \int_x^\infty \frac{\sin\lambda(t-x)}{\lambda}V(t)E(t, \lambda)\,dt, \quad \operatorname{Im}\lambda \leq 0, \tag{1.3.4}$$

which is equivalent to the differential equation (1.1.2) with the boundary condition

$$\lim_{x\to\infty} e^{i\lambda x}E(x, \lambda) = I.$$

We formally seek a solution of (1.3.4) in the form (1.3.2). Substitution of the expression (1.3.2) for $E(x, \lambda)$ in (1.3.4) yields

$$\int_x^\infty K(x, t)e^{-i\lambda t}\,dt = \int_x^\infty \frac{\sin\lambda(s-x)}{\lambda}V(s)e^{-i\lambda s}\,ds +$$

$$+ \int_x^\infty V(s)\,ds \int_s^\infty \frac{\sin\lambda(s-x)}{\lambda}e^{-i\lambda u}K(s, u)\,du = J_1 + J_2. \tag{1.3.5}$$

We now transform the integrals on the right-hand side of (1.3.5) first making use of the formulas

$$\frac{\sin\lambda(s-x)}{\lambda}e^{-i\lambda s}=\frac{1}{2}\int_{x}^{2s-x}e^{-i\lambda t}\,dt,$$

$$\frac{\sin\lambda(s-x)}{\lambda}e^{-i\lambda u}=\frac{1}{2}\int_{u+x-s}^{u+s-x}e^{-i\lambda t}\,dt,$$

and then interchanging the orders of integration.

This yields

$$J_1=\frac{1}{2}\int_{x}^{\infty}V(s)\,ds\int_{x}^{2s-x}e^{-i\lambda t}\,dt=\int_{x}^{\infty}e^{-i\lambda t}\left[\frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}V(s)\,ds\right]dt;$$

$$J_2=\frac{1}{2}\int_{x}^{\infty}V(s)\,ds\int_{s}^{\infty}K(s,u)\,du\int_{u+x-s}^{u+s-x}e^{-i\lambda t}\,dt=$$

$$=\int_{x}^{\infty}e^{-i\lambda t}\left[\frac{1}{2}\int_{x}^{\frac{x+t}{2}}V(s)\,ds\int_{t+x-s}^{t+s-x}K(s,u)\,du+\right.$$

$$\left.+\frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}V(s)\,ds\int_{s}^{t+s-x}K(s,u)\,du\right]dt.$$

The change of the order of integration in the integral J_1 is easily justified using the condition (1.3.1); as to J_2, we perform the interchange formally at this point and justify the operation below.

Substituting the expressions found for J_1 and J_2 into (1.3.5) and making use of the uniqueness of the Fourier integral representation, we arrive at the following integral equation for the matrix $K(x,t)$:

$$K(x,t)=\frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}V(s)\,ds+\frac{1}{2}\int_{x}^{\frac{x+t}{2}}V(s)\,ds\int_{t+x-s}^{t+s-x}K(s,u)\,du+$$

$$+\frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}V(s)\,ds\int_{s}^{t+s-x}K(s,u)\,du\qquad(0<x\leq t).\quad(1.3.6)$$

Applying the method of successive approximations, we now show that a solution to this equation exists. To this end, we define

$$K_0(x, t) = \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} V(s)\, ds,$$

$$K_m(x, t) = \frac{1}{2} \int_{x}^{\frac{x+t}{2}} V(s)\, ds \int_{t+x-s}^{t+s-x} K_{m-1}(s, u)\, du +$$

$$+ \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} V(s)\, ds \int_{s}^{t+s-x} K_{m-1}(s, u)\, du,$$

and we obtain the following estimates (since $x \leq t$):

$$| K_0(x, t) | \leq \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} | V(s) |\, ds = \frac{1}{2} \sigma\left(\frac{x+t}{2}\right);$$

$$| K_1(x, t) | \leq \frac{1}{2} \int_{x}^{\frac{x+t}{2}} | V(s) |\, ds \int_{t+x-s}^{t+s-x} \frac{1}{2} \sigma\left(\frac{s+u}{2}\right) du +$$

$$+ \frac{1}{2} \int_{\frac{x+t}{2}}^{\infty} | V(s) |\, ds \int_{s}^{t+s-x} \frac{1}{2} \sigma\left(\frac{s+u}{2}\right) du \leq$$

$$\leq \frac{1}{2} \sigma\left(\frac{x+t}{2}\right) \int_{x}^{\infty} s\, | V(s) |\, ds = \frac{1}{2} \sigma\left(\frac{x+t}{2}\right) \sigma_1(x).$$

By mathematical induction, we then conclude that

$$| K_m(x, t) | \leq \frac{1}{2} \sigma\left(\frac{x+t}{2}\right) \frac{\sigma_1^m(x)}{m\,!} \qquad (m = 0, 1, 2, \ldots).$$

From this estimate, if follows that the series

$$\sum_{m=0}^{\infty} K_m(x, t)$$

converges uniformly in the interval $0 < a \leq x \leq t$ for any $a > 0$, and that the inequality (1.3.3) holds for its sum $K(x, t)$. It is easy to see that the matrix $K(x, t)$ obtained in this way satisfies the integral equation (1.3.6).

Finally,

$$\int_x^\infty \sigma(u)\, du = \int_x^\infty du \int_u^\infty |V(t)|\, dt = \int_x^\infty |V(t)|\, dt \int_x^t du =$$

$$= \int_x^\infty (t-x)\,|V(t)|\, dt \leq \sigma_1(x).$$

and applying (1.3.3), we find that

$$\int_x^\infty |K(x, t)|\, dt \leq \frac{1}{2} e^{\sigma_1(x)} \int_x^\infty \sigma\left(\frac{x+t}{2}\right) dt =$$

$$= e^{\sigma_1(x)} \int_x^\infty \sigma(u)\, du \leq e^{\sigma_1(x)} \sigma_1(x) \qquad (1.3.7)$$

We can now justify the interchange of the order of integration in the integral J_2 (see (1.3.5)). In fact, it is permissible to apply Fubini's Theorem here because by (1.3.7)

$$\int_x^\infty s\,|V(s)|\, ds \int_s^\infty |K(s, u)|\, du \leq e^{\sigma_1(x)} \sigma_1^2(x) < \infty$$

for any value of $x > 0$ (and for $x = 0$, as well, in the event that $\sigma_1(0) < \infty$).

This completes the proof that the expression (1.3.2), with a kernel $K(x, t)$ satisfying (1.3.6), is a solution of (1.1.2). From the representation (1.3.2) and the inequality (1.3.7), it obviously follows that $E(x, \lambda)$ is regular in the half-plane $\operatorname{Im} \lambda < 0$ for each fixed value of x (and for $x = 0$, as well, in the event that $\sigma_1(0) < \infty$) and is continuous there up to and including the real axis.

REMARK. Equation (1.3.6) implies that $K(x, t)$ has continuous partial derivatives with respect to each of its arguments where, in particular,

$$K_x(x, t) = -\frac{1}{4} V\left(\frac{x+t}{2}\right) - \frac{1}{2}\int_x^{\frac{x+t}{2}} V(s)K(s, t+x-s)\,ds -$$
$$-\frac{1}{2}\int_x^{\infty} V(s)K(s, t+s-x)\,ds \qquad (0 < x \leq t). \tag{1.3.8}$$

An analogous expression holds for $K_t(x, t)$. By means of the estimate (1.3.3), it follows from (1.3.8) that

$$\left|K_x(x, t)\right| \leq \frac{1}{4}\left|V\left(\frac{x+t}{2}\right)\right| + \frac{1}{4}e^{\sigma_1(x)}\sigma\left(\frac{x+t}{2}\right)\int_x^{\frac{x+t}{2}} |\, V(s)\,|\, ds +$$
$$+\frac{1}{4}e^{\sigma_1(x)}\int_x^{\infty} |\, V(s)\,|\, \sigma\left(\frac{2s+t-x}{2}\right) ds \leq \frac{1}{4}\left|V\left(\frac{x+t}{2}\right)\right| +$$
$$+\frac{1}{2}e^{\sigma_1(x)}\sigma\left(\frac{x+t}{2}\right)\sigma(x). \tag{1.3.9}$$

If $V(x)$ is differentiable, $K(x, t)$ then has second partial derivatives, and from (1.3.6) it is possible to derive the following equation for $K(x, t)$:

$$\frac{\partial^2}{\partial x^2}K(x, t) - V(x)K(x, t) = \frac{\partial^2}{\partial t^2}K(x, t). \tag{1.3.10}$$

Also immediately following from (1.3.6) is the relation

$$K(x, x) = \frac{1}{2}\int_x^{\infty} V(t)\,dt, \tag{1.3.11}$$

which will be important in the later development.

Conversely, it is easy to show, that if $K(x, t)$ satisfies equation (1.3.10), the relation (1.3.11), and the conditions

$$\lim_{x+t\to\infty} K_x(x, t) = \lim_{x+t\to\infty} K_t(x, t) = 0, \tag{1.3.12}$$

then it is a solution of equation (1.3.6). Thus the matrix $E(x, \lambda)$

which, $K(x, t)$ determines by means of formula (1.3.2)] is a solution of equation (1.1.2) with $V(x) = -2\ dK(x, x)/dx$.

LEMMA 1.3.1: *If* $\sigma_1(0) < \infty$, *then*

$$\lim_{x \to +0} xE'(x, \lambda) = 0 \qquad (\operatorname{Im} \lambda \leq 0)$$

and

$$\lim_{x \to +0} x\dot{E}'(x, \lambda) = 0 \qquad (\operatorname{Im} \lambda < 0),$$

the dot denoting differentiation with respect to λ.

Proof: The successive differentiation of (1.3.2) with respect to x and λ yields

$$E'(x, \lambda) = -i\lambda e^{-i\lambda x} I - K(x, x) e^{-i\lambda x} + \\ + \int_x^\infty K_x(x, t) e^{-i\lambda t}\, dt \qquad (\operatorname{Im} \lambda \leq 0), \tag{1.3.13}$$

$$\dot{E}'(x, \lambda) = -ie^{-i\lambda x} I - \lambda x e^{-i\lambda x} I + ixK(x, x) e^{-i\lambda x} - \\ - i \int_x^\infty K_x(x, t) t e^{-i\lambda t}\, dt \qquad (\operatorname{Im} < 0). \tag{1.3.14}$$

The validity of these two expressions is proved in the same way, and we shall give the reasoning only for the latter one.

Let

$$c = \frac{1}{2} e^{\sigma_1(0)}.$$

By means of the inequality (1.3.9) and with x and λ such that

$$0 < x_0 \leq x, \qquad \operatorname{Im} \lambda = -\mu_0 < 0,$$

we have

$$\left| \int_N^\infty t K_x(x, t) e^{-i\lambda t}\, dt \right| \leq \int_N^\infty t \left| K_x(x, t) \right| e^{-\mu_0 t}\, dt \leq$$

$$\leq \frac{1}{4} e^{-\mu_0 N} \int_N^\infty t \left| V\left(\frac{x+t}{2}\right) \right| dt + c\sigma(x) \int_N^\infty t\sigma\left(\frac{x+t}{2}\right) e^{-\mu_0 t}\, dt \leq$$

$$\leq e^{-\mu_0 N} \int_{\frac{1}{2}N}^\infty u \left| V(u) \right| du + c\sigma(x_0)\sigma\left(\frac{N}{2}\right) \int_N^\infty t e^{-\mu_0 t}\, dt.$$

This estimate shows that the integral on the right-hand side of (1.3.14) is uniformly convergent for x and λ in the above region for any positive quantities x_0 and μ_0. This implies the validity of (1.3.14) for $x > 0$ and $\operatorname{Im} \lambda < 0$.

Now, using the inequalities (1.3.3) and (1.3.9), we obtain the following three estimates:

$$| K(x, x) | \le c\sigma(x);$$

for $\operatorname{Im} \lambda \le 0$,

$$\left| \int_x^\infty K_x(x, t) e^{-i\lambda t}\, dt \right| \le \frac{1}{4} \int_x^\infty \left| V\left(\frac{x+t}{2}\right) \right| dt + c\sigma(x) \int_x^\infty \sigma\left(\frac{x+t}{2}\right) dt =$$

$$= \frac{1}{2} \int_x^\infty | V(u) |\, du + 2c\sigma(x) \int_x^\infty \sigma(u)\, du \le \left[\frac{1}{2} + 2c\sigma_1 (0) \right] \sigma(x);$$

for $\operatorname{Im} \lambda = -\mu < 0$,

$$\left| \int_x^\infty tK_x(x, t) e^{-i\lambda t}\, dt \right| \le \frac{1}{4} \int_x^\infty t \left| V\left(\frac{x+t}{2}\right) \right| dt +$$

$$+ c\sigma(x) \int_x^\infty t\sigma\left(\frac{x+t}{2}\right) e^{-\mu t}\, dt \le \int_x^\infty u \,| V(u) |\, du + c\sigma(x) \int_x^\infty t\sigma\left(\frac{t}{2}\right) e^{-\mu t}\, dt \le$$

$$\le \sigma_1 (0) + 2c\mu^{-1}\sigma_1 (0)\, \sigma(x),$$

since

$$t\sigma\left(\frac{t}{2}\right) = t \int_{\frac{t}{2}}^\infty | V(s) |\, ds \le 2 \int_{\frac{t}{2}}^\infty s\, | V(s) |\, ds = 2\sigma_1\left(\frac{t}{2}\right) \le 2\sigma_1 (0).$$

From these estimates and (1.3.13) and (1.3.14), it is apparent that to complete the proof of the lemma it suffices to show that

$$\lim_{x \to +0} x\sigma(x) = 0.$$

But for $0 < x < 1$, we have

$$x\sigma(x) = x \int_{x}^{\infty} |V(s)|\, ds = x\Big[\int_{x}^{\sqrt{x}} |V(s)|\, ds + \int_{\sqrt{x}}^{\infty} |V(s)|\, ds\Big] \leq$$

$$\leq \int_{x}^{\sqrt{x}} s\,|V(s)|\, ds + \sqrt{x} \int_{\sqrt{x}}^{\infty} s\,|V(s)|\, ds,$$

and the right-hand side of this inequality tends to zero as $x \to 0$

Lemma 1.3.2: *Let the potential* $V(x)$ *in equation* (1.1.2) *be Hermitian and satisfy condition* (1.3.1). *Then for all positive values of* x

$$W\{E^*(x,\lambda),\, E(x,\lambda)\} = \begin{cases} -2i\lambda I & \text{for } \operatorname{Im}\lambda = 0 \\ 0 & \text{for } \operatorname{Re}\lambda = 0, \quad \operatorname{Im}\lambda < 0, \end{cases}$$

$$W\{E^*(x,\lambda),\, E(x,-\lambda)\} = 0 \text{ for } \operatorname{Im}\lambda = 0.$$

Proof: If $V(x)$ is Hermitian and $Y(x)$ and $Z(x)$ are any two solutions of the equation (1.1.2), then when $\operatorname{Im}\lambda^2 = 0$, we have

$$Y^{*\prime\prime}(x) + \lambda^2 Y^*(x) = Y^*(x)V(x),$$

$$Z''(x) + \lambda^2 Z(x) = V(x)Z(x).$$

Hence,

$$Y^*(x)Z''(x) - Y^{*\prime\prime}(x)Z(x) = 0$$

which implies that

$$W\{Y^*(x)Z(x)\} = Y^*(x)Z'(x) - Y^{*\prime}(x)Z(x)$$

is independent of x. By means of this and the fact that

$$e^{i\lambda x}E(x,\lambda) \to I, \quad e^{i\lambda x}E'(x,\lambda) \to -i\lambda I$$

as $x \to \infty$, the relations of the lemma result upon letting $x \to \infty$ in their respective left-hand members.

§ 4. A Fundamental System of Solutions with a Prescribed Behavior at Infinity for the Case $\lambda \neq 0$

Theorem 1.4.1: *If for some* $\delta \geq 0$

$$\int^{\infty} x^{1+\delta}\, |V(x)|\, dx < \infty, \tag{1.4.1}$$

then for any non-zero value of λ *in the halfplane* $\operatorname{Im}\lambda \leq 0$, *the equation*

(1.1.2) *has a fundamental system of solutions* $E(x, \lambda)$ *and* $E^{(1)}(x, \lambda)$ *with the following asymptotic behavior:*

$$E(x, \lambda) = e^{-i\lambda x}[I + o(x^{-1-\delta})], \quad E'(x, \lambda) = -i\lambda e^{-i\lambda x}[I + o(x^{-1-\delta})],$$

$$(x \to \infty)$$

$$E^{(1)}(x, \lambda) = e^{i\lambda x}[I + o(x^{-1-\delta})], \quad E^{(1)\prime}(x, \lambda) = i\lambda e^{i\lambda x}[I + o(x^{-1-\delta})].$$

For $\operatorname{Im} \lambda \leq 0$, *these solutions can be represented in the form*

$$E(x, \lambda) = e^{-i\lambda x}\left[I + \frac{1}{2i\lambda}\int_x^\infty V(t)\, dt + A(x, \lambda)\right],$$

$$E'(x, \lambda) = -i\lambda e^{-i\lambda x}\left[I + \frac{1}{2i\lambda}\int_x^\infty V(t)\, dt + B(x, \lambda)\right],$$

$$E^{(1)}(x, \lambda) = e^{i\lambda x}\left[I - \frac{1}{2i\lambda}\int_x^\infty V(t)\, dt + A^{(1)}(x, \lambda)\right],$$

$$E^{(1)\prime}(x, \lambda) = i\lambda e^{i\lambda x}\left[I - \frac{1}{2i\lambda}\int_x^\infty V(t)\, dt + B^{(1)}(x, \lambda)\right],$$

where the matrices $A(x, \lambda)$, $B(x, \lambda)$, $A^{(1)}(x, \lambda)$, *and* $B^{(1)}(x, \lambda)$ *satisfy conditions analogous to* (1.4.1).

Proof: According to Theorem 1.3.1, the condition (1.4.1) assures the existence of the special solution $E(x, \lambda)$ of equation (1.1.2) determined by the integral equation (1.3.4) and given by (1.3.2). Differentiating (1.3.4) with respect to x, we obtain

$$E'(x, \lambda) = -i\lambda e^{-i\lambda x}I - \int_x^\infty \cos \lambda(t-x) V(t) E(t, \lambda)\, dt. \qquad (1.4.2)$$

By means of (1.3.4), (1.4.2), the condition (1.4.1), and the boundedness of $e^{i\lambda x} E(x, \lambda)$, we can show rather easily that $E(x, \lambda)$ and $E'(x, \lambda)$ have the asymptotic behavior at infinity stated in the theorem.

If we use (1.3.2) and (1.3.4), we further obtain

$$E(x,\lambda) = e^{-i\lambda x}\left[I+\frac{1}{2i\lambda}\int_x^\infty e^{i\lambda t}V(t)E(t,\lambda)\,dt-\frac{e^{2i\lambda x}}{2i\lambda}\int_x^\infty e^{-i\lambda t}V(t)E(t,\lambda)\,dt\right] =$$

$$= e^{-i\lambda x}\left[I+\frac{1}{2i\lambda}\int_x^\infty V(t)dt+A(x,\lambda)\right],$$

where

$$A(x,\lambda) = \frac{1}{2i\lambda}\int_x^\infty V(t)\,dt\int_t^\infty K(t,s)e^{-i\lambda(s-t)}\,ds-$$

$$-\frac{e^{2i\lambda x}}{2i\lambda}\int_x^\infty e^{-2i\lambda t}V(t)\left[I+\int_t^\infty K(t,s)e^{-i\lambda(s-t)}\,ds\right]dt.$$

In a similar way, it follows from equations (1.3.2) and (1.4.2) that

$$E'(x,\lambda) = -i\lambda e^{-i\lambda x}\left[I+\frac{1}{2i\lambda}\int_x^\infty V(t)dt+B(x,\lambda)\right],$$

where

$$B(x,\lambda) = \frac{1}{2i\lambda}\int_x^\infty V(t)\,dt\int_t^\infty K(t,s)e^{-i\lambda(s-t)}\,ds+$$

$$+\frac{e^{2i\lambda x}}{2i\lambda}\int_x^\infty e^{-2i\lambda t}V(t)\left[I+\int_t^\infty K(t,s)e^{-i\lambda(s-t)}\,ds\right]dt.$$

Now, let $\operatorname{Im}\lambda = -\mu < 0$. Then (see (1.3.3))

$$\left|\int_x^\infty V(t)\,dt\int_t^\infty K(s,t)e^{-i\lambda(s-t)}\,ds\right| \leq$$

$$\leq\frac{1}{2}\,e^{\sigma_1(x)}\int_x^\infty |\,V(t)\,|\,\sigma(t)\,dt\int_t^\infty e^{-\mu(s-t)}\,ds\leq\frac{1}{2\mu}\,e^{\sigma_1(x)}\sigma^2(x).$$

Moreover, by (1.4.1) and the fact that $\int_a^\infty \sigma(x)\,dx \le \sigma_1(a) < \infty$,

$$\int_a^\infty x^{1+\delta}\sigma^2(x)\,dx \le \int_a^\infty \sigma(x)\,dx \int_x^\infty t^{1+\delta}\,|V(t)|\,dt < \infty \qquad (a>0).$$

Also, for $b > a > 0$,

$$\int_a^b x^{1+\delta}e^{2\mu x}\,dx \int_x^\infty e^{-2\mu t}\,|V(t)|\,dt = x^{1+\delta}\int_x^\infty e^{-2\mu t}\,|V(t)|\,dt \cdot \frac{e^{2\mu x}}{2\mu}\Big|_a^b -$$

$$-\frac{1}{2\mu}\int_a^b \left[(1+\delta)x^{\delta}e^{2\mu x}\int_x^\infty e^{-2\mu t}\,|V(t)|\,dt - x^{1+\delta}\,|V(x)|\right]dx \le$$

$$\le \frac{b^{1+\delta}e^{2\mu b}}{2\mu}\int_b^\infty e^{-2\mu t}\,|V(t)|\,dt + \frac{1}{2\mu}\int_a^b x^{1+\delta}\,|V(x)|\,dx \le$$

$$\le \frac{1}{2\mu}\int_b^{2b} t^{1+\delta}\,|V(t)|\,dt + \frac{b^{1+\delta}e^{-2\mu b}}{2\mu}\int_{2b}^\infty |V(t)|\,dt + \frac{1}{2\mu}\int_a^b x^{1+\delta}\,|V(x)|\,dx.$$

This result together with (1.4.1) yields the inequality

$$\int_a^\infty x^{1+\delta}e^{2\mu x}\,dx \int_x^\infty e^{-2\mu t}\,|V(t)|\,dt < \infty \qquad (a > 0).$$

From these estimates it is easily seen that

$$\int_a^\infty x^{1+\delta}\,|A(x, \lambda)|\,dx < \infty, \qquad \int_a^\infty x^{1+\delta}\,|B(x, \lambda)|\,dx < \infty \qquad (a>0).$$

We now proceed to construct the matrix $E^{(1)}(x, \lambda)$, and for $\operatorname{Im}\lambda = 0$, we define

$$E^{(1)}(x, \lambda) = E(x, -\lambda) \qquad (\operatorname{Im}\lambda = 0,\ \lambda \ne 0).$$

$E(x, -\lambda)$ is a solution of (1.1.2) and has the required asymptotic behavior at infinity.

If $\operatorname{Im}\lambda = -\mu < 0$, we define $E^{(1)}(x, \lambda)$ for sufficiently large values of $x > 0$ as the solution of the integral equation

$$E^{(1)}(x, \lambda) = e^{i\lambda x}I - \frac{e^{i\lambda x}}{2i\lambda}\int_x^\infty e^{-i\lambda t}V(t)E^{(1)}(t, \lambda)\,dt - \\ - \frac{e^{-i\lambda x}}{2i\lambda}\int_h^x e^{i\lambda t}V(t)E^{(1)}(t, \lambda)\,dt. \tag{1.4.3}$$

After the substitution

$$E^{(1)}(x, \lambda) = e^{i\lambda x}F(x, \lambda)$$

this equation becomes

$$F(x, \lambda) = I - \frac{1}{2i\lambda}\int_x^\infty V(t)F(t, \lambda)\,dt - \frac{e^{-2i\lambda x}}{2i\lambda}\int_h^x e^{2i\lambda t}V(t)F(t, \lambda)\,dt.$$

It is easy to show by the method of successive approximations that a solution to this equation exists in the interval $h \leq x < \infty$ provided h is positive and is such that

$$\frac{1}{2|\lambda|}\int_h^\infty |V(t)|\,dt = q < 1.$$

In this connection, it is found that

$$|F(x, \lambda)| \leq \frac{1}{1-q} \qquad (x \geq h),$$

and this implies that for $x \geq 2h$

$$|F(x, \lambda) - I| \leq \frac{1}{2|\lambda|}\int_x^\infty |V(t)|\,|F(t, \lambda)|\,dt + \\ + \frac{e^{-2\mu x}}{2|\lambda|}\int_h^x e^{2\mu t}|V(t)|\,|F(t, \lambda)|\,dt \leq \text{const}\left[\int_{\frac{x}{2}}^\infty |V(t)|\,dt + \right. \\ \left. + e^{-\mu x}\int_h^{\frac{x}{2}} |V(t)|\,dt\right] \leq \text{const}\left[\sigma\left(\frac{x}{2}\right) + e^{-\mu x}\right]. \tag{1.4.4}$$

From this and (1.4.1), we thus conclude that

$$E^{(1)}(x, \lambda) = e^{i\lambda x}F(x, \lambda) = e^{i\lambda x}[I+o(x^{-1-\delta})] \qquad (x \to \infty).$$

Now, differentiating (1.4.3) with respect to x, we obtain

$$E^{(1)\prime}(x, \lambda) = i\lambda e^{i\lambda x}I - \frac{e^{i\lambda x}}{2}\int_x^\infty e^{-i\lambda t}V(t)E^{(1)}(t, \lambda)\,dt + \\ + \frac{e^{-i\lambda x}}{2}\int_h^x e^{i\lambda t}V(t)E^{(1)}(t, \lambda)\,dt, \tag{1.4.5}$$

and just as in the above, we find that

$$E^{(1)\prime}(x, \lambda) = i\lambda e^{i\lambda x}[I+o(x^{-1-\delta})] \qquad (x \to \infty).$$

From (1.4.3) and (1.4.5), if further follows that

$$E^{(1)}(x, \lambda) = e^{i\lambda x}\left[I - \frac{1}{2i\lambda}\int_x^\infty V(t)\,dt + A^{(1)}(x, \lambda)\right],$$

$$E^{(1)\prime}(x, \lambda) = i\lambda e^{i\lambda x}\left[I - \frac{1}{2i\lambda}\int_x^\infty V(t)\,dt + B^{(1)}(x, \lambda)\right],$$

where

$$A^{(1)}(x, \lambda) = -\frac{1}{2i\lambda}\int_x^\infty V(t)[F(t, \lambda)-I]\,dt - \frac{e^{-2i\lambda x}}{2i\lambda}\int_h^x e^{2i\lambda t}V(t)F(t, \lambda)\,dt,$$

$$B^{(1)}(x, \lambda) = -\frac{1}{2i\lambda}\int_x^\infty V(t)[F(t, \lambda)-I]\,dt + \\ + \frac{e^{-2i\lambda x}}{2i\lambda}\int_h^x e^{2i\lambda t}V(t)F(t, \lambda)\,dt.$$

Applying the inequality (1.4.4), we have

$$\left|\int_x^\infty V(t)[F(t, \lambda)-I]\,dt\right| \leq \text{const}\int_x^\infty |V(t)|\left[\sigma\left(\frac{t}{2}\right)+e^{-\mu t}\right]dt \leq \\ \leq \text{const}\left[\sigma\left(\frac{x}{2}\right)+e^{-\mu x}\right]\sigma(x),$$

and the refore

$$\int_h^\infty x^{1+\delta} \left| \int_x^\infty V(t)[F(t, \lambda) - I]\, dt \right| dx \leq$$

$$\leq \text{const} \int_h^\infty \left[\sigma\left(\frac{x}{2}\right) + e^{-\mu x}\right] dx \int_x^\infty t^{1+\delta} |V(t)|\, dt < \infty.$$

Now, for $b > h$

$$\int_h^b x^{1+\delta} e^{-2\mu x}\, dx \int_h^x e^{2\mu t} |V(t)|\, dt = x^{1+\delta} \int_h^x e^{2\mu t} |V(t)|\, dt \left.\frac{e^{-2\mu x}}{-2\mu}\right|_h^b +$$

$$+ \frac{1}{2\mu} \int_h^b \left[(1+\delta) x^\delta e^{-2\mu x} \int_h^x e^{2\mu t} |V(t)|\, dt + x^{1+\delta} |V(x)|\right] dx \leq$$

$$\leq \frac{1}{2\mu} \int_h^b \left[(1+\delta) x^\delta e^{-2\mu x} \int_h^x e^{2\mu t} |V(t)|\, dt + x^{1+\delta} |V(x)|\right] dx,$$

and for $b \geq 2h$

$$\int_h^b x^\delta e^{-2\mu x}\, dx \int_h^x e^{2\mu t} |V(t)|\, dt = \text{const} + \int_{2h}^b x^\delta e^{-2\mu x}\, dx \int_h^x e^{2\mu t} |V(t)|\, dt \leq$$

$$\leq \text{const} + \int_{2h}^b x^\delta e^{-\mu x}\, dx \int_h^{\frac{x}{2}} |V(t)|\, dt + \int_{2h}^b x^\delta\, dx \int_{\frac{x}{2}}^x |V(t)|\, dt,$$

so that by (1.4.1),

$$\int_h^\infty x^{1+\delta} e^{-2\mu x}\, dx \int_h^x e^{2\mu t} |V(t)|\, dt < \infty.$$

A straightforward computation using the above results shows that

$$\int^\infty x^{1+\delta} |A^{(1)}(x, \lambda)|\, dx < \infty, \qquad \int^\infty x^{1+\delta} |B^{(1)}(x, \lambda)|\, dx < \infty.$$

and this completes the proof of the theorem.

§ 5. A fundamental System of Solutions with a Prescribed Behavior at Infinity for $\lambda = 0$

THEOREM 1.5.1: *If the condition* (1.4.1) *is satisfied, then the equation* (1.1.2) *with* $\lambda = 0$, *i.e.,*

$$Y'' = V(x)Y \tag{1.5.1}$$

has a fundamental system of solutions $E(x, 0)$ *and* $E^{(1)}(x)$ *with the following asymptotic behavior:*

$$\begin{aligned} E(x,0) &= I + o(x^{-\delta}), & E'(x,0) &= o(x^{-1-\delta}),\\ E^{(1)}(x) &= x[I+o(x^{-\delta})], & E^{(1)\prime}(x) &= I + o(x^{-\delta}). \end{aligned} \qquad (x \to \infty)$$

Proof: The required special solution $E(x, 0)$ of (1.5.1) satisfies the integral equation (see (1.3.4))

$$E(x,0) = I + \int_x^\infty (t-x)V(t)E(t,0)\,dt,$$

which upon differentiation with respect to x yields

$$E'(x,0) = -\int_x^\infty V(t)E(t,0)\,dt.$$

From these two equations and the condition (1.4.1), it immediately follows that $E(x, 0)$ has the asymptotic behavior at infinity stated in the theorem.

$E^{(1)}(x)$ is defined as the solution of an integral equation whose form depends on what the value of the exponent δ is in the inequality (1.4.1). Namely, if $0 \leq \delta < 1$, then we choose

$$E^{(1)}(x) = xI - x\int_x^\infty V(t)E^{(1)}(t)\,dt - \int_h^x tV(t)E^{(1)}(t)\,dt, \tag{1.5.2}$$

for $h \leq x < \infty$, where h is a positive quantity such that

$$\int_h^\infty t\,|V(t)|\,dt < 1;$$

if $\delta \geq 1$, then

$$E^{(1)}(x) = xI + \int_x^\infty (t-x)V(t)E^{(1)}(t)\,dt, \qquad 0 < x < \infty. \tag{1.5.3}$$

It is not difficult to show by means of the method of successive approximations, that (1.5.2) and (1.5.3) have solutions in the specified intervals, and that the matrix $E^{(1)}(x)$ obtained in this way satisfies equation (1.5.1). The differentiation of (1.5.2) and (1.5.3) with respect to x then yields the following identical result:

$$E^{(1)\prime}(x) = I - \int_x^\infty V(t)E^{(1)}(t)\,dt. \tag{1.5.4}$$

Let $0 \leq \delta < 1$. From (1.5.2), it follows that the matrix $x^{-1}\,E^{(1)}(x)$ is bounded in the interval $h \leq x < \infty$, and by virtue of this and (1.4.1), we find from (1.5.4) that

$$E^{(1)\prime}(x) = I + o(x^{-\delta}) \qquad (x \to \infty).$$

To find the asymptotic behavior of $E^{(1)}(x)$, we start from the following relation which results from integration by parts:

$$\int_h^x tV(t)E^{(1)}(t)\,dt = -\int_t^\infty sV(s)\,dsE^{(1)}(t)\,\Big|_h^x + \int_h^x \Big[\int_t^\infty sV(s)\,ds\Big]E^{(1)\prime}(t)\,dt.$$

By (1.4.1) and the boundedness of the matrices $x^{-1}E^{(1)}(x)$ and $E^{(1)\prime}(x)$, we have

$$-\int_t^\infty sV(s)\,dsE^{(1)}(t)\,\Big|_h^x = o(x^{1-\delta}) \qquad (x \to \infty);$$

$$\Big|\int_h^x \Big[\int_t^\infty sV(s)\,ds\Big]E^{(1)\prime}(t)\,dt\,\Big| \leq \text{const} \int_h^x dt \int_t^\infty s\,|V(s)|\,ds =$$

$$= \text{const}\Big[\int_h^x s\,|V(s)|\,ds \int_h^s dt + \int_x^\infty s\,|V(s)|\,ds \int_h^x dt\Big] \leq$$

$$\leq \text{const}\Big[\int_h^x s^2\,|V(s)|\,ds + x\int_x^\infty s\,|V(s)|\,ds\Big] \leq$$

$$\leq \text{const}\; x^{1-\delta}\Big[x^{-\frac{1-\delta}{2}} \int_h^{\sqrt{x}} s^{1+\delta}\,|V(s)|\,ds + \int_{\sqrt{x}}^\infty s^{1+\delta}\,|V(s)|\,ds\Big] =$$

$$= o(x^{1-\delta}) \qquad (x \to \infty).$$

These estimates applied to (1.5.2) show that

$$E^{(1)}(x) = x[I + o(x^{-\delta})] \qquad (x \to \infty).$$

By making use of (1.5.3) and (1.5.4), we can easily satisfy ourselves that $E^{(1)}(x)$ and $E^{(1)\prime}(x)$ have the same asymptotic behavior for $\delta \geq 1$ as they have for $0 \leq \delta < 1$.

LEMMA 1.5.1: *If the potential* $V(x)$ *in equation* (1.5.1) *is Hermitian and satisfies both* (1.2.1) *and* (1.4.1), *then the solution* $G(x, 0)$ *of this equation* (see Theorem 1.2.1) *has the following asymptotic behavior:*

$$G(x, 0) = xE^*(0) + A + o(x^{1-\delta}), \quad G'(x, 0) = E^*(0) + o(x^{-\delta}) \quad (x \to \infty),$$

where $E(0) = E(0,0)$ *and* A *is a constant matrix.*

Proof: By Theorem 1.5.1, we have

$$\begin{aligned} G(x, 0) &= E(x, 0)A + E^{(1)}(x)B, \\ G'(x, 0) &= E'(x, 0)A + E^{(1)\prime}(x)B, \end{aligned} \tag{1.5.5}$$

in which A and B are constant matrices, and therefore it is clear that we merely have to show that $B = E^*(0)$.

Let $Y(x)$ and $Z(x)$ be arbitrary solutions of equation (1.5.1). Then since $V(x)$ is Hermitian, it follows that the Wronskian

$$W\{Y^*(x), Z(x)\} = Y^*(x)Z'(x) - Y^{*\prime}(x)Z(x),$$

is independent of x. In particular,

$$W\{E^*(x, 0),\ E(x, 0)\} = \lim_{x\to\infty} W\{E^*(x, 0),\ E(x, 0)\} = 0,$$

$$W\{E^*(x, 0),\ E^{(1)}(x)\} = \lim_{x\to\infty} W\{E^*(x, 0),\ E^{(1)}(x)\} = I.$$

Therefore, if we premultiply the first equation of (1.5.5) by $E^{*\prime}(x, 0)$ and the second by $E^*(x, 0)$ and then subtract the second from the first, we obtain

$$E^*(x, 0)\, G'(x, 0) - E^{*\prime}(x, 0)\, G(x, 0) = B.$$

We now let x approach zero. Since $G(x, 0) \sim xI$ and $G'(x, 0) \sim I$ as $x \to 0$, it follows from Lemma 1.3.1 that $B = E^*(0)$, q.e.d.

CHAPTER II

The Spectrum and Scattering Matrix for the Boundary-Value Problem without Singularities

§ 1. The Point Spectrum

In what follows, we shall consider the boundary-value problem:

$$\left.\begin{aligned} y_j'' + \lambda^2 y_j &= \sum_{k=1}^{n} v_{jk}(x) y_k \quad 0 < x < \infty \\ y_j(0) &= 0 \qquad (j = 1, 2, \ldots n) \end{aligned}\right\} \tag{2.1.1}$$

under the assumption that the potential $V(x) = ||v_{jk}||_1^n$ is a Hermitian matrix such that

$$\int_0^\infty x \,|\, V(x) \,|\, dx < \infty \tag{2.1.2}$$

We shall agree to call it the *non-singular problem,* and we begin by studying its point spectrum.

Consider the special solution $E(x, \lambda)$ of

$$Y'' + \lambda^2 Y = V(x) Y, \qquad 0 < x < \infty \tag{2.1.3}$$

(see Theorem 1.3.1), and let

$$E(\lambda) = E(0, \lambda). \tag{2. 1.4}$$

The matrix $E(\lambda)$ is regular in the half-plane $\operatorname{Im} \lambda < 0$ and is continous there up to and including the real axis.

Suppose that for some $\lambda = \lambda_0$, with $\operatorname{Im} \lambda_0 < 0$, $\det E(\lambda_0) = 0$. This implies the existence of a non-zero vector a such that

$$E(\lambda_0) a = 0.$$

But then the vector function $E(x, \lambda_0)a$ is a solution of the system

(1.1.1) vanishing at $x = 0$ and decreasing exponentially as $x \to \infty$. Thus, λ_0^2 is an eigenvalue of the boundary-value problem (2.1.1). Since this problem is self-adjoint, its eigenvalues are all real, and λ_0 is therefore of the form $-i\mu_0$, with $\mu_0 > 0$.

Thus, if $\det E(\lambda)$ is a regular function of λ in the lower half-plane, then it can have zeros there only on the imaginary axis, and the squares of these zeros will be eigenvalues for the boundary-value problem (2.1.1).

It is easy to show that the problem (2.1.1) has no eigenvalues other than the squares of the zeros of $\det E(\lambda)$. In fact, by Theorems 1.4.1 and 1.5.1, every solution of (1.1.1) can be represented in the form

$$y(x, \lambda) = E(x, \lambda)a + E^{(1)}(x, \lambda)b,$$

in which a and b are constant vectors. The components of $y(x, \lambda)$ are square summable over the interval $(0, \infty)$ if and only if $\operatorname{Im} \lambda \neq 0$, where $b = 0$ in the particular case $\operatorname{Im} \lambda < 0$, i.e.,

$$y(x, \lambda) = E(x, \lambda)a \qquad (\operatorname{Im} \lambda < 0).$$

In order for λ_0^2, with $\operatorname{Im} \lambda_0 < 0$, to be an eigenvalue of the problem (2.1.1) it is necessary for a non-zero vector a to exist such that

$$E(0, \lambda_0)a = E(\lambda_0)a = 0,$$

and this is equivalent to $\det E(\lambda_0) = 0$.

Now, according to formulas (2.1.4) and (1.3.2), $E(\lambda) \to I$ when $|\lambda| \to \infty$ $(\operatorname{Im} \lambda \leq 0)$,[1] so that the zeros of $\det E(\lambda)$ form a bounded set with at most one limit point possible, namely $\lambda = 0$. Therefore if $\det E(0) \neq 0$, then the number of zeros of $\det E(\lambda)$ is finite, which means that the boundary-value problem (2.1.1) has a finite number of eigenvalues.

It turns out that the set of eigenvalues is finite even if $\det E(0) = 0$, i.e., the following theorem holds.

---

[1] When $\operatorname{Im} \lambda < 0$, this statement is an immediate consequence of the expression

$$E(\lambda) = I + \int_0^\infty K(0, t)e^{-i\lambda t}\, dt.$$

On the other hand, if $\operatorname{Im} \lambda = 0$, the Riemann—Lebesgue Theorem is then applicable since by (1.3.3) and (2.1.2), $K(0,t)$ is summable over the entire interval $(0,\infty)$.

THEOREM 2.1.1: *The non-singular boundary-value problem* (2.1.1) *has a finite number of eigenvalues (which are negative).*

Proof:[2] Let $L^2_{(n)}(\alpha, \beta)$ denote the Hilbert space of vector functions $f(x) = \{f_1(x), ..., f_n(x)\}$ with components square summable over the interval (α, β), and with a scalar product defined by

$$(f, g)_{<\alpha,\, \beta>} = \int_\alpha^\beta \sum_{k=1}^n f_k(x)\bar{g}_k(x)\, dx.$$

Let

$$l[y] = -y'' + V(x)y,$$

where y is a column vector with the components $y_1(x), .y_2(x), \ldots$ $y_n(x)$, and let us consider the self-adjoint operator L defined in $L^2_{(n)}(0, \infty)$ by $l[y]$ and the boundary condition $y(0) = 0$. The domain of the operator L, which we denote by D_L, is to consist of all functions $y(x) \in L^2_{(n)}(0, \infty)$ satisfying the following conditions:

(1) $y'(x)$ exists and is absolutely continuous in each finite interval $(0, a)$;

(2) $l[y] \in L^2_{(n)}(0, \infty)$;

(3) $y(0) = 0$.

For all $y(x) \in D_L$, we have by definition

$$Ly = l[y].$$

The boundary-value problem (2.1.1) is clearly equivalent to the operator equation

$$Ly = \lambda^2 y,$$

and our problem becomes that of showing that L has a finite number of eigenvalues.

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[2] The basis of the proof is the method of decomposition of operators due to Glazman [11]. If in addition to (2.1.2), the condition

$$\int_0^\infty x^2 \,|\, V(x) \,|\, dx < \infty \qquad (!)$$

is satisfied, the proof of the theorem becomes considerably simpler and the ideas of operator theory need not be resorted to. In fact, the theorem being proved is a simple consequence of the formula

$$E^{-1}(\lambda) = \lambda^{-1}N + o(\lambda^{-1}) \qquad (\lambda \to 0,\ \mathrm{Im}\, \lambda \leq 0),$$

which can be derived by elementary means when the conditions (2.1.2) and (!) hold.

To this end, we choose some number $\alpha > 0$, and we introduce two additional self-adjoint operators L_1 and L_2 as follows:

L_1 is the operator which is defined in $L^2_{(n)}(0, \alpha)$ by $l[y]$ and the boundary conditions $y(0) = y(\alpha) = 0$; and

L_2 is the operator which is defined in $L^2_{(n)}(\alpha, \infty)$ by $l[y]$ and the boundary condition $y(\alpha) = 0$.

The respective domains of these operators, D_{L_1} and D_{L_2}, are formed analogously to D_L, and for all $y(x) \in D_{L_i}$, we have by definition

$$L_i y = l[y] \qquad (i=1,2).$$

Let us show that L_2 has no eigenvalues when α is a sufficiently large positive quantity. By means of the same reasoning used at the beginning of this section, we find first that all of the eigenvalues of L_2 are the squares of the zeros of det $E(\alpha, \lambda)$, which, if they exist, are of the form $\lambda = -i\mu$, $\mu > 0$. Now according to (1.3.2),

$$E(\alpha, -i\mu) = e^{-\mu\alpha} I + \int_\alpha^\infty K(\alpha, t) e^{-\mu t}\, dt.$$

Moreover,

$$\left| \int_\alpha^\infty K(\alpha, t) e^{-\mu t} \right| dt \leq e^{-\mu\alpha} \int_\alpha^\infty | K(\alpha, t) |\, dt = e^{-\mu\alpha} o\,(1)$$

as $\alpha \to \infty$, for by applying the inequality (1.3.3), we obtain

$$\int_\alpha^\infty | K(\alpha, t) |\, dt \leq c \int_\alpha^\infty \sigma\left(\frac{\alpha+t}{2}\right) dt = 2c \int_\alpha^\infty \sigma(u)\, du \leq 2c\sigma_1(\alpha),$$

where $c = e^{\sigma_1(0)}/2$.

Hence, it follows that the function det $E(\alpha, -i\mu)$, $\mu \geq 0$, will have no zeros if α is sufficiently large.

Choose and fix a positive value of α so that this condition has been satisfied; then the operator L_2 will have no point spectrum. Furthermore, since the continuous spectrum of L_2 is situated on the positive axis when (2.1.1) holds, L_2 is a non-negative operator, i.e., for every function $y(x)$ belonging to D_{L_2}

$$(L_2 y, y) = (l[y], y)_{<\alpha, \infty>} \geq 0.$$

Let us now examine the operator L_1, and let us show that it has a finite number of negative eigenvalues.

In fact, it can easily be seen that the eigenvalues of this operator are the squares of the zeros of det $G(\alpha, \lambda)$ (see Theorem 1.2.1), and because these eigenvalues are real, the negative ones are of the form $-\mu^2$, where μ is a positive quantity such that

$$\det G(\alpha, -i\mu) = 0.$$

Now, the substitution of $\lambda = -i\mu$ and $x = \alpha$ in (1.2.8) yields

$$|\,\mu e^{-\mu\alpha} G(\alpha, -i\mu) - \sinh \mu\alpha\, e^{-\mu\alpha} I\,| \to 0$$

as $\mu \to \infty$. But the function det $G(\alpha, \lambda)$ is even in λ, and the last result shows that its zeros, which lie on the imaginary axis, form a bounded set in the λ-plane; since det $G(\alpha, \lambda)$ is an entire function, the number of zeros must be finite.

Let m be the sum of the dimensions of the subspaces corresponding to each of the negative eigenvalues of the operator L_1, and let p be the number of eigenvalues of L. It then follows that

$$p \leq m+n. \tag{2.1.5}$$

Let us assume the contrary, i.e., $p > m+n$. Hence, there exist at least $m+n+1$ linearly independent eigenvectors of the operator L corresponding to these eigenvalues:

$$y^{(1)}(x),\ y^{(2)}(x), \ldots, y^{(m+n+1)}(x).$$

From these eigenvectors, it is obviously possible to form $m+1$ linearly independent linear combinations

$$z^{(1)}(x),\ z^{(2)}(x), \ldots, z^{(m+1)}(x),$$

which vanish at $x = \alpha$. The vectors $z^{(i)}(x)$ $(i = 1,2,\ldots,m+1)$ thus satisfy the conditions

$$z^{(i)}(0) = z^{(i)}(\alpha) = 0 \qquad (i = 1, 2, \ldots, m+1)$$

and therefore belong to both D_{L_1} and D_{L_2}.

It is now possible to form a non-trivial linear combination of the $z^{(i)}(x)$,

$$u(x) = \sum_{i=1}^{m+1} \gamma_i z^{(i)}(x),$$

which is orthogonal in the metric of $L^2_{(n)}(0, \alpha)$ to all of the eigen subspaces of the operator L_1 corresponding to its negative eigenvalues.

For the vector $u(x)$, we have the two relations

$$(L_1 u, u) = (l[u], u)_{<0,\,\alpha>} \geq 0,$$

$$(L_2 u, u) = (l[u], u)_{<\alpha,\,\infty>} \geq 0$$

(the second one being a consequence of the non-negativeness of the operator L_2).

However, since $u(x)$ is by definition a linear combination of the eigenvectors of the operator L, and since L has only negative eigenvalues, it follows that

$$(Lu, u) = (l[u], u)_{<0, \infty>} < 0.$$

Finally, noting that

$$(l[u], u)_{<0, \infty>} = (l[u], u)_{<0, \alpha>} + (l[u], u)_{<\alpha, \infty>},$$

we arrive at a contradiction because the left-hand side of this equation is negative whereas the right-hand side is non-negative.

Remark. Define

$$V_\beta(x) = \begin{cases} V(x), & 0 < x < \beta \\ 0, & x \geq \beta \end{cases}$$

and consider the boundary-value problem (2.1.1) with $V(x)$ replaced by $V_\beta(x)$. The proof of the theorem clearly shows that when $\beta \geq \alpha$ the number of eigenvalues of the new problem having $V_\beta(x)$ as its potential will satisfy the same inequality (2.1.5). In other words, the number of eigenvalues of the boundary-value problem in which $V(x)$ is replaced by $V_\beta(x)$, is uniformly bounded for all sufficiently large values of $\beta \geq \alpha$.

§ 2. Properties of the Matrix $E^{-1}(\lambda)$

From the results of the preceding section, it follows that the matrix function $E^{-1}(\lambda)$ is regular in the half-plane $\operatorname{Im} \lambda < 0$, with the possible exception of a finite number of points on the imaginary axis where $\det E(\lambda) = 0$, and, hence, where $E^{-1}(\lambda)$ has poles.

In order to investigate the character of the singular points of $E^{-1}(\lambda)$, we shall need the following lemma.

Lemma 2.2.1: *Let $A(z)$ be a square matrix, which is regular in the circle $|z| < 1$, such that* $\det A(0) = 0$ *and* $\det A(z) \neq 0$ *for* $0 < |z| < 1$.

Then the inverse matrix $A^{-1}(z)$ has a simple pole at $z = 0$ if and only if the relations

$$A(0)a = 0,$$
$$A(0)b + A'(0) = 0, \tag{2.2.1}$$

where a and b are constant vectors, imply that $a = 0$.

Proof:[3] It is clear that $A^{-1}(z)$ is regular everywhere inside the circle $|z| < 1$ except at $z = 0$ where it has a pole of some order m. We may therefore write

$$A^{-1}(z) = z^{-m}N_{-m}+z^{-m+1}N_{-m+1}+ \dots,$$
$$A(z) = A_0+zA_1 \dots,$$

where $A_0 = A(0)$, $A_1 = A'(0)$, etc.

The matrices N_r obviously satisfy the following equations:

$$\sum_{s=0}^{r} A_s N_{-s+r-m} = \sum_{s=0}^{r} N_{-s+r-m} A_s = \delta_{rm} I \tag{2.2.2}$$

$$(r=0,\ 1,\ 2, \dots),$$

with

$$\delta_{rm} = \begin{cases} 1, & \text{if } r=m \\ 0, & \text{if } r\neq m. \end{cases}$$

Suppose that $m = 1$, i.e., $A^{-1}(z)$ has a simple pole at $z = 0$. Then from (2.2.2) there results

$$A_0N_{-1}=N_{-1}A_0=0, \tag{2.2.3}$$
$$A_0N_0+A_1N_{-1} = N_{-1}A_1+N_0A_0 = I,$$

in which the matrix $N_{-1} \neq 0$.

Postmultiplying the second equation of (2.2.3) by the vector a and making use of the first equation of (2.2.1), we obtain

$$a=N_{-1}A_1a.$$

By the second equation of (2.2.1), we find that

$$a = -N_{-1}A_0b,$$

and this is equal to zero as a consequence of (2.2.3).

The necessity is thus proved.

Suppose now that from the relations (2.2.1), it follows that $a = 0$. Using this fact, we find successively from (2.2.2) that $N_{-m} = 0$, $N_{-m+1} = 0$, etc., until we arrive at the equations (2.2.3). Thus, $A^{-1}(z)$ has only a simple pole at $z = 0$. This completes the proof of the sufficiency of the criterion.

---

[3] The proof of this lemma and that of Theorem 2.2.1, which follows have been taken from the article of Newton and Jost [1].

THEOREM 2.2.1: *All the singularities of the matrix function* $E^{-1}(\lambda)$ *in the half-plane* $\mathrm{Im}\ \lambda < 0$ *are simple poles.*

Proof: We begin by considering the differential equation

$$E''(x, \lambda)+\lambda^2 E(x, \lambda) = V(x)E(x, \lambda). \tag{2.2.4}$$

Differentiating it with respect to λ and then taking the complex conjugate transpose of both sides of the resulting equation, we obtain [4]

$$\dot{E}^{*\prime\prime}(x, \lambda)+\bar{\lambda}^2\dot{E}^*(x, \lambda)+2\bar{\lambda}E^*(x, \lambda) = E^*(x, \lambda)V(x).$$

Assuming that $\mathrm{Re}\ \lambda = 0$ and $\mathrm{Im}\,\lambda < 0$, we postmultiply this equation by $E(x, \lambda)$ and subtract the result from (2.2.4) after first premultiplying it by $\dot{E}^*(x, \lambda)$. Hence,

$$\dot{E}^*(x, \lambda)E''(x, \lambda)-\dot{E}^{*\prime\prime}(x, \lambda)E(x, \lambda) = -2\lambda E^*(x, \lambda)E(x, \lambda)$$

($\mathrm{Re}\,\lambda = 0$ and $\mathrm{Im}\,\lambda < 0$). If we integrate both sides of this equation from x to ∞ taking into consideration that $E(x, \lambda)$, $E'(x, \lambda)$, and $\dot{E}'(x, \lambda)$ tend to zero as $x \to \infty$ when $\mathrm{Im}\,\lambda < 0$, we obtain

$$\dot{E}^*(x, \lambda)E'(x, \lambda)-\dot{E}^{*\prime}(x, \lambda)E(x, \lambda) = 2\lambda \int_x^\infty E^*(t, \lambda)E(t, \lambda)\,dt \tag{2.2.5}$$

$$(\mathrm{Re}\,\lambda=0, \quad \mathrm{Im}\,\lambda<0).$$

Let $\lambda_0 = -i\,\mu_0$, with μ_0 positive, be a pole of $E^{-1}(\lambda)$. This implies that $\det E(\lambda_0) = 0$, and hence that there exists a non-zero vector a such that

$$E(\lambda_0)a=0 \tag{2.2.6}$$

(vectors are to be regarded as column matrices).

Thus the solution $E(x, \lambda_0)a$ of the system (2.2.4) vanishes at $x = 0$ when $\lambda = \lambda_0$. Hence, by Theorem 1.2.1, it follows that

$$E(x, \lambda_0)a=G(x, \lambda_0)a_1,$$

where a_1 is some vector, and therefore,

$$\lim_{x\to 0} E'(x, \lambda_0)a=E'(\lambda_0)a$$

exists (the right-hand side merely denotes the limit).

---

[4] The dot denotes differentiation with respect to λ.

Making use of this and Lemma 1.3.1, we then have [5]

$$\lim_{x\to 0}\{a^*\dot{E}^{*\prime}(x,\lambda)E(x,\lambda_0)a\}=\lim_{x\to 0}\{[xa^*\dot{E}^{*\prime}(x,\lambda_0)][x^{-1}E(x,\lambda_0)a]\}=$$

$$=\lim_{x\to 0}\{[xa^*\dot{E}^{*\prime}(x,\lambda_0)][E'(\theta x,\lambda_0)a]\}=0.$$

Upon setting $\lambda = \lambda_0$ in (2.2.5), multiplying this equation on the left by a^* and on the right by a and then letting x tend to zero, we obtain

$$a^*\dot{E}^*(\lambda_0)E'(\lambda_0)a=2\lambda_0\int_0^\infty [E(t,\lambda_0)a]^*[E(t,\lambda_0)a]\,dt\neq 0. \qquad (2.2.7)$$

Let us now assume that the vector a satisfies not only the condition (2.2.6) but also the equation

$$E(\lambda_0)b+\dot{E}(\lambda_0)a = 0. \qquad (2.2.8)$$

where b is some other vector.

Taking the complex conjugate transpose of both sides of the matrix equation (2.2.8) and then postmultiplying the result by $E'(\lambda_0)a$, we obtain

$$b^*E^*(\lambda_0)E'(\lambda_0)a+a^*E^*(\lambda_0)E'(\lambda_0)a = 0.$$

But by Lemma 1.3.2 and equation (2.2.6),

$$b^*E^*(\lambda_0)E'(\lambda_0)a=b^*E^{*\prime}(\lambda_0)E(\lambda_0)a=0.$$

and therefore

$$a^*E^*(\lambda_0)E'(\lambda_0)a=0.$$

This is a contradiction to (2.2.7).

This shows that no non-zero vector a exists satisfying conditions (2.2.6) and (2.2.8) simultaneously. But then by Lemma 2.2.1, $E^{-1}(\lambda)$ has a simple pole at λ_0 and the theorem is thus proved.

We shall now prove one further lemma which can be used to describe the behavior of $E^{-1}(\lambda)$ on the real axis.

LEMMA 2.2.2: *For any value of* $x \geq 0$, *the matrix* $E(x,\lambda)$ *is nonsingular when* λ *is real and non-zero.*

---

[6] If a complex matrix $A(t) = \| a_{jk}(t)+ib_{jk}(t) \|$ is differentiable, then

$$A(t+h)-A(t) = h \,\| a'_{jk}(t+\theta'_{jk}h)+ib'_{jk}(t+\theta''_{jk}h) \|,$$

where $0 < \theta'_{jk} < 1$, $0 < \theta''_{jk} < 1$. For brevity, the right-hand side of this equation will be denoted by $hA'(t+\theta h)$.

Proof: By Lemma 1.3.2, the relation

$$E^*(x,\lambda)E'(x,\lambda)-E^{*\prime}(x,\lambda)E(x,\lambda) = -2i\lambda I$$

holds for $\operatorname{Im}\lambda = 0$.

Multiplying this equation on the right by a vector a and on the left by a^* yields

$$a^*[E^*(x,\lambda)E'(x,\lambda)-E^{*\prime}(x,\lambda)E(x,\lambda)]a = -2i\lambda a^*a.$$

From this, we immediately conclude that if $E(x,\lambda)a = 0$ and $\lambda \neq 0$, then $a^*a = 0$ and hence $a = 0$, q.e.d.

Since $E(\lambda)$ is continuous in the half-plane $\operatorname{Im}\lambda \leq 0$, Lemma 2.2.2 implies that $E^{-1}(\lambda)$ is bounded and continuous for all real values of λ with the possible exception of $\lambda = 0$. The behavior of $E^{-1}(\lambda)$ in the neighborhood of $\lambda = 0$ will be investigated in §4.

§ 3. The Scattering Matrix

Throughout this section, λ will be assumed to be real and non-zero. Consider the solution $G(x,\lambda)$ of (2.1.3) satisfying the initial conditions

$$G(0,\lambda)=0, \qquad G'(0,\lambda)=I$$

(see Theorem 1.2.1).

Applying Theorem 1.4.1 and recalling that $E^{(1)}(x,\lambda) = E(x,-\lambda)$ when λ is real and non-zero (see the proof of Theorem 1.4.1), we can write

$$G(x,\lambda) = E(x,\lambda)A+E(x,-\lambda)B,$$
$$G'(x,\lambda) = E'(x,\lambda)A+E'(x,-\lambda)B,$$

where A and B are matrices not depending on x. Premultiplying the first of these by $E^{*\prime}(x,\lambda)$ and the second by $E^*(x,\lambda)$ and then subtracting the results, we have as a consequence of Lemma 1.3.2

$$A = \frac{1}{2i\lambda}[E^{*\prime}(x,\lambda)G(x,\lambda)-E^*(x,\lambda)G'(x,\lambda)],$$

in which x may be any positive quantity. In a similar way, we find that

$$B = \frac{1}{2i\lambda}[E^*(x,-\lambda)G'(x,\lambda)-E^{*\prime}(x,-\lambda)G(x,\lambda)].$$

Since $G(x, \lambda) \sim x\,I$ as $x \to 0$, Lemma 1.3.1 gives

$$\lim_{x \to 0} E^{*\prime}(x, \lambda)\, G(x, \lambda) = \lim_{x \to 0} E^{*\prime}(x, -\lambda) G(x, \lambda) = 0.$$

Therefore, letting x tend to zero in the expressions for A and B, we arrive at

$$A = -\frac{1}{2i\lambda} E^*(\lambda), \qquad B = \frac{1}{2i\lambda} E^*(-\lambda).$$

As a consequence,

$$G(x, \lambda) = \frac{1}{2i\lambda}[E(x, -\lambda)E^*(-\lambda) - E(x, \lambda)E^*(\lambda)],$$

and upon setting $x = 0$ in this, we obtain the important result

$$E(-\lambda)E^*(-\lambda) = E(\lambda)E^*(\lambda). \tag{2.3.1}$$

On the basis of Lemma 2.2.2, the expression for $G(x, \lambda)$ can be written in the form

$$G(x, \lambda) = \frac{1}{2i\lambda}[E(x, -\lambda) - E(x, \lambda)S(-\lambda)]E^*(-\lambda), \tag{2.3.2}$$

where

$$S(\lambda) = E^*(-\lambda)\,[E^*(\lambda)]^{-1}.$$

By (2.3.1), this may also be expressed as

$$S(\lambda) = E^*(-\lambda)\,[E^*(\lambda)]^{-1} = E^{-1}(-\lambda)E(\lambda), \tag{2.3.3}$$

from which it easily follows that $S(\lambda)$ is a *unitary matrix*, i.e.,

$$S(\lambda)S^*(\lambda) = S^*(\lambda)S(\lambda) = I,$$

and, moreover, that

$$S(-\lambda) = S^*(\lambda). \tag{2.3.4}$$

The matrix $S(\lambda)$ is called the *scattering matrix*. In Chapter IV, we shall explain how $S(\lambda)$ determines the asymptotic form at infinity of the normalized eigenfunctions corresponding to the continuous spectrum of the problem (2.1.1).

THEOREM 2.3.1: *The matrix $I—S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$,* i.e.,

$$I—S(\lambda) = \int_{-\infty}^{\infty} F_s(t)e^{-i\lambda t}\, dt, \tag{2.3.5}$$

where

$$F_s(t) = F_s^{(1)}(t) + F_s^{(2)}(t),$$

$|F_s^{(1)}(t)| \in L(-\infty, \infty)$, $|F_s^{(2)}(t)| \in L^2(-\infty, \infty)$, *and both functions are bounded for all real values of* t. [6]

Proof: Let us denote by $\widetilde{L}$ the linear manifold of functions $\widetilde{f}(\lambda)$, each of which is the Fourier transform of a function in $L(-\infty, \infty)$, i.e., which can be represented as

$$\widetilde{f}(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t}\, dt, \quad f(t) \in L(-\infty, \infty).$$

The product of any two functions in $\widetilde{L}$ is also known to belong to $\widetilde{L}$.

Now consider the elements of the matrix $I - S(\lambda)$, namely, the functions

$$\widetilde{f}_{ij}(\lambda) = \delta_{ij} - s_{ij}(\lambda),$$

where

$$\| \delta_{ij} \|_1^n = I, \quad \| s_{ij}(\lambda) \|_1^n = S(\lambda) = E^{-1}(-\lambda) E(\lambda) \tag{2.3.6}$$

(see (2.3.3)). For $\operatorname{Im} \lambda = 0$, (1.3.2) gives

$$E(\lambda) = \| \delta_{ij} + \widetilde{k}_{ij}(\lambda) \|_1^n, \quad E(-\lambda) = \| \delta_{ij} + \widetilde{k}_{ij}(-\lambda) \|_1^n, \tag{2.3.7}$$

in which

$$\| \widetilde{k}_{ij}(\lambda) \|_1^n = \int_0^{\infty} K(0, t) e^{-i\lambda t}\, dt.$$

Since

$$\int_0^{\infty} | K(0, t) |\, dt \leq \frac{1}{2} e^{\sigma_1(0)} \int_0^{\infty} \sigma\left(\frac{t}{2}\right) dt = e^{\sigma_1(0)} \cdot \sigma_1(0) < \infty$$

by (1.3.3), each of the functions $\widetilde{k}_{ij}(\lambda)$ and $\widetilde{k}_{ij}(-\lambda)$ belongs to $\widetilde{L}$.

---

[6] The right-hand side of (2.3.5) is equal to

$$\int_{-\infty}^{\infty} F_s^{(1)}(t) e^{-i\lambda t}\, dt + \int_{-\infty}^{\infty} F_s^{(2)}(t) e^{-i\lambda t}\, dt,$$

the first integral converging absolutely and the second converging in the mean.

The fact that the absolute value of a matrix belongs to $L(-\infty, \infty)$ or $L^2(-\infty, \infty)$ is equivalent to each of its elements belonging to $L(-\infty, \infty)$ or $L^2(-\infty, \infty)$.

From (2.3.7), it further follows that

$$\det E(-\lambda) = 1+p(\lambda), \tag{2.3.8}$$

where $p(\lambda)$ is a sum of products of the $\tilde{k}_{ij}(-\lambda)$, and therefore $p(\lambda)\in\tilde{L}$. In exactly the same way, it is found that the cofactors of the elements of $\det E(-\lambda)$, which we denote by $\varepsilon_{ij}(-\lambda)$, are expressible as

$$\varepsilon_{ij}(-\lambda) = \delta_{ij}+p_{ij}(\lambda), \tag{2.3.9}$$

where $p_{ij}(\lambda) \in \tilde{L}$ $(i, j = 1, 2, \ldots, n)$.

Thus, equations (2.3.6) through (2.3.9) imply that

$$s_{ij}(\lambda) = \frac{1}{\det E(-\lambda)} \sum_{\nu=1}^{n} \varepsilon_{\nu i}(-\lambda)\,[\delta_{\nu j}+\tilde{k}_{\nu j}(\lambda)] =$$
$$= \frac{1}{1+p(\lambda)} \sum_{\nu=1}^{n} [\delta_{\nu i}+p_{\nu i}(\lambda)]\,[\delta_{\nu j}+\tilde{k}_{\nu j}(\lambda)],$$

which clearly can be written in the form

$$s_{ij}(\lambda) = [1+p(\lambda)]^{-1}[\delta_{ij}+\varphi_{ij}(\lambda)], \tag{2.3.10}$$

where $\varphi_{ij}(\lambda) \in \tilde{L}$ $(i, j = 1, 2, \ldots, n)$.

Suppose now that $\det E(0) \neq 0$, so that $\det E(-\lambda) = 1+p(\lambda) \neq 0$ on the entire closed real axis (see Lemma 2.2.2). Then by a well-known theorem of Wiener [12],

$$[1+p(\lambda)]^{-1} = 1+q(\lambda), \qquad q(\lambda) \in \tilde{L}.$$

Therefore, from (2.3.10), we have

$$\tilde{f}_{ij}(\lambda) = \delta_{ij}-s_{ij}(\lambda) = \delta_{ij}-[1+q(\lambda)]\,[\delta_{ij}+\varphi_{ij}(\lambda)] =$$
$$= -\varphi_{ij}(\lambda)-q(\lambda)\,\delta_{ij}-q(\lambda)\varphi_{ij}(\lambda).$$

Hence, we conclude that $\tilde{f}_{ij}(\lambda) \in \tilde{L}$ $(i, j = 1, 2, \ldots, n)$. In other words, in case $\det E(0) \neq 0$, the relation (2.3.5) holds and $|F_s(t)| \in L(-\infty, \infty)$.

Now let $\det E(0) = 0$. Since as $|\lambda| \to \infty$

$$\det E(-\lambda) = 1+p(\lambda) \to 1$$

(see (1.3.2)), it follows that $p(\lambda) \to 0$ when $|\lambda| \to \infty$. Therefore, a positive number l can be found such that

$$|p(\lambda)| < 1 \quad \text{for} \quad |\lambda| \geq l.$$

Consider the function $h(\lambda)$ defined by

$$h(\lambda) = \begin{cases} 1, & \text{if} \quad |\lambda| < l \\ -|\lambda|+l+1, & \text{if} \quad l \leq |\lambda| \leq l+1 \\ 0, & \text{if} \quad |\lambda| > l+1 \end{cases}$$

and set

$$p(\lambda)[1-h(\lambda)] = r(\lambda).$$

Since $p(\lambda)$ and $h(\lambda)$ both belong to $\widetilde{L}$, $r(\lambda)$ also belongs to $\widetilde{L}$. Moreover,

$$r(\lambda) = p(\lambda) \quad \text{for} \quad |\lambda| > l+1$$

and

$$|r(\lambda)| < 1, \qquad -\infty < \lambda < \infty.$$

Now set $S^{(1)}(\lambda) = \| s_{ij}^{(1)}(\lambda) \|_1^n$ where

$$s_{ij}^{(1)}(\lambda) = [1+r(\lambda)]^{-1}[\delta_{ij}+\varphi_{ij}(\lambda)] \qquad (i,\ j = l,\ 2, \ldots, n). \quad (2.3.11)$$

Since $1+r(\lambda) \neq 0$ on the entire closed real axis, applying Wiener's Theorem, we find that

$$s_{ij}^{(1)}(\lambda) = \delta_{ij}+\psi_{ij}(\lambda), \quad \psi_{ij}(\lambda) \in \widetilde{L}, \qquad (2.3.12)$$
$$(i,\ j=1,\ 2, \ldots, n).$$

Hence, it follows in particular, that $s_{ij}^{(1)}(\lambda)$ is bounded on the entire real axis, and from (2.3.10) and (2.3.11) we have

$$S^{(1)}(\lambda) = S(\lambda) \quad \text{for} \quad |\lambda| > l+1.$$

Noting further that

$$|s_{ij}(\lambda)| \leq 1 \qquad (\operatorname{Im} \lambda = 0, \lambda \neq 0)$$

because $S(\lambda)$ is a unitary matrix, we conclude that the matrix

$$S^{(2)}(\lambda) = S(\lambda) - S^{(1)}(\lambda) \qquad (2.3.13)$$

vanishes for $|\lambda| > l+1$ and that its elements $s_{ij}^{(2)}(\lambda)$ are bounded for all real values of $\lambda \neq 0$. Therefore, the quantities $s_{ij}^{(2)}(\lambda)$ are the Fourier transforms of bounded functions in $L^2(-\infty, \infty)$.

Thus, for the case where $\det E(0) = 0$, (2.3.12) and (2.3.13) show that

$$\widetilde{f}_{ij}(\lambda) = \delta_{ij} - s_{ij}(\lambda) = -\psi_{ij}(\lambda) - s_{ij}^{(2)}(\lambda)$$

with $\psi_{ij}(\lambda) \in \widetilde{L}$ and $s_{ij}^{(2)}(\lambda)$ the Fourier transform of a bounded function in $L^2(-\infty, \infty)$.

It still remains to prove that $F_s(t)$ is Hermitian. This follows easily, for by the property (2.3.4) of $S(\lambda)$ and the fact that $\operatorname{Im} \lambda = 0$, (2.3.5) yields

$$\int_{-\infty}^{\infty} F_s^*(t)e^{-i\lambda t}\,dt = \Big(\int_{-\infty}^{\infty} F_s(t)e^{i\lambda t}\,dt\Big)^* = I-S^*(-\lambda) = I-S(\lambda)$$
$$= \int_{-\infty}^{\infty} F_s(t)e^{-i\lambda t}\,dt.$$

As a consequence of the uniqueness of the Fourier transform, we conclude that $F_s^*(t) = F_s(t)$.

Remark. If $\det E(0) = 0$, then the system (1.1.1) with $\lambda = 0$ has a non-trivial bounded solution which vanishes at $x = 0$ (see Theorem 1.5.1). In this event, we shall say that the boundary-value problem (2.1.1) has a *virtual level.* Theorem 1.5.1 also implies that the problem (2.1.1) has no virtual level if $\det E(0) \neq 0$.

As we have just seen, if no virtual level exists, then the elements of the matrix $F_s(t)$ belong to $L(-\infty, \infty)$.

In Chapter V (Theorem 5.6.2), we shall prove that even if $\det E(0) = 0$, the elements of $F_s(t)$ still belong to $L(-\infty, \infty)$.

The proof of this considerably more refined theorem must be postponed to the very end of our investigation of the non-singular boundary-value problem.

§ 4. Behavior of the Matrix $E^{-1}(\lambda)$ in the Neighborhood of $\lambda = 0$

To supplement the results of §2, we shall next study the behavior of $E^{-1}(\lambda)$ in the neighborhood of $\lambda = 0$. It is clear that this question only arises when there exists a virtual level ($\det E(0) = 0$) since if $\det E(0) \neq 0$, $E^{-1}(\lambda)$ is continuous everywhere on the real axis including the point $\lambda = 0$.

Lemma 2.4.1: *The matrix $\lambda E^{-1}(\lambda)$ is bounded in the neighborhood of $\lambda = 0$.*

Proof: [7] Let

$$V_\beta(x) = \begin{cases} V(x), & \text{if} \quad 0 < x < \beta \\ 0, & \text{if} \quad x \geq \beta \end{cases}$$

---

[7] If in addition to (2.1.2), the condition

$$\int_0^{\infty} x^2 \, | V(x) | \, dx < \infty \qquad (!)$$

is satisfied, then the lemma being proved is trivial (see footnote 2 on p. 39) and denote by $E_\beta(x, \lambda)$ and $G_\beta(x, \lambda)$ those solutions of

$$Y'' + \lambda^2 Y = V_\beta(x) Y$$

which are analogous to the respective solutions $E(x, \lambda)$ and $G(x, \lambda)$ of (2.1.3).

From the equation

$$G(x, \lambda) = \frac{\sin \lambda x}{\lambda} I + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} V(t) G(t, \lambda)\, dt \qquad (2.4.1)$$

and the analogous integral equation for $G_\beta(x, \lambda)$, it follows that

$$G_\beta(x, \lambda) = G(x, \lambda) \quad \text{for} \quad 0 \leq x < \beta. \qquad (2.4.2)$$

Furthermore, by definition, we have

$$E_\beta(x, \lambda) = e^{i\lambda x} I + \int_x^\infty K_\beta(x, t) e^{-i\lambda t}\, dt, \qquad (2.4.3)$$

where $K_\beta(x, t)$ satisfies the integral equation obtained by replacing $V(x)$ in (1.3.6) by $V_\beta(x)$. Making use of this equation and (1.3.6) one can easily show that

$$\lim_{\beta \to \infty} \int_x^\infty | K(x, t) - K_\beta(x, t) |\, dt = 0$$

uniformly for $x \geq 0$. This result and (1.3.2) and (2.4.3) then imply that

$$\lim_{\beta \to \infty} E_\beta(x, \lambda) - E(x, \lambda) \qquad (2.4.4)$$

uniformly for $x \geq 0$ and λ in the half-plane $\operatorname{Im} \lambda \leq 0$.

From the equation defining $K_\beta(x, t)$, it is also evident that $K_\beta(x, t) = 0$ when $(x+t)/2 > \beta$, and therefore for each fixed value of $x \geq 0$, the integral on the right-hand side of (2.4.3) is over a finite interval. Hence, for each value of $x \geq 0$, $E_\beta(x, \lambda)$ is an entire function of λ.

By Theorems 2.1.1 and 2.2.1 and the remark of Theorem 2.1.1, $E_\beta^{-1}(\lambda) = E_\beta^{-1}(0, \lambda)$ is regular everywhere in the half-plane $\operatorname{Im} \lambda < 0$, except for a finite number of points on the imaginary axis where it has simple poles, and for all sufficiently large values of β, the number of poles is uniformly bounded, say, by N. In what follows, β will be considered to be such a sufficiently large quantity and no particular mention of this will be made each time. On the basis of

Lemma 2.2.2 (which obviously holds for $E_\beta(x, \lambda)$ as well), $E_\beta^{-1}(\lambda)$ is also regular for all real values of $\lambda \neq 0$.

Now let x_0 be an any non-negative quantity and suppose that $\beta > x_0$. Consider the relation analogous to (2.3.2) which, by virtue of (2.4.2), can be expressed in the form

$$2i\lambda G(x_0, \lambda)[E_\beta^{-1}(-\lambda)]^* = E_\beta(x_0, -\lambda) - E_\beta(x_0, \lambda)S_\beta(-\lambda)$$
$$(\mathrm{Im}\ \lambda = 0,\ \lambda \neq 0),$$

or

$$2i\lambda E_\beta^{-1}(\lambda)G^*(x_0, -\lambda) = E_\beta^*(x_0, \lambda) - S_\beta(-\lambda)E_\beta^*(x_0, -\lambda)$$
$$(\mathrm{Im}\ \lambda = 0,\ \lambda \neq 0).$$

Since $S_\beta(-\lambda)$ is a unitary matrix and $E_\beta(x_0, \lambda)$ is uniformly bounded for all positive values of β and $-\infty < \lambda < \infty$,[8] it follows that

$$|2i\lambda E_\beta^{-1}(\lambda)G^*(x_0, -\lambda)| = |E_\beta^*(x_0, \lambda) - S_\beta(-\lambda)E_\beta^*(x_0, -\lambda)| < k$$

for all real values of $\lambda \neq 0$. Here, k is a constant not depending on β or λ. Now, recalling that $E_\beta(\lambda)$ and $G(x_0, -\lambda)$ are entire functions of λ, we see that the function

$$\Phi_\beta(\lambda) = 2i\lambda E_\beta^{-1}(\lambda)G^*(x_0, -\overline{\lambda})$$

is meromorphic in the whole plane and cannot have a pole (in other words, is regular) at $\lambda = 0$. As a result,

$$|\Phi_\beta(\lambda)| \leq k \qquad (-\infty < \lambda < \infty). \tag{2.4.5}$$

Upon letting $\beta \to \infty$ and noting that

$$\lim_{\beta \to \infty} \Phi_\beta(\lambda) = \Phi(\lambda) = 2i\lambda E^{-1}(\lambda)G^*(x_0, -\lambda)$$

exists for all real values of $\lambda \neq 0$ because of (2.4.4) and Lemma 2.2.2, we obtain

$$|\Phi(\lambda)| = |2i\lambda E^{-1}(\lambda)G^*(x_0, -\lambda)| \leq k \quad (\mathrm{Im}\ \lambda = 0,\ \lambda \neq 0). \tag{2.4.6}$$

It is then possible to find a positive number r such that $E^{-1}(\lambda)$ will have no singularities on the semicircle

$$|\lambda| = r, \quad \mathrm{Im}\ \lambda < 0, \tag{2.4.7}$$

---

[8] This follows immediately from (2.4.3) since the estimate

$$|K_\beta(x, t)| \leq c\sigma\left(\frac{x+t}{2}\right), \quad c = \frac{1}{2}e^{\sigma_1(0)}$$

holds for any positive value of β [cf. (1.3.3)].

and in the domain D interior to the semicircle. The function $\Phi(\lambda)$ is bounded on this semicircle, i.e.,

$$|\Phi(\lambda)| = |2i\lambda E^{-1}(\lambda)G^*(x_0, -\bar{\lambda})| < k \quad (|\lambda| = r, \ \operatorname{Im}\lambda < 0), \tag{2.4.8}$$

where the constant k may obviously be assumed to be the same as the one in the inequality (2.4.6). Since $E_\beta^{-1}(\lambda) \to E^{-1}(\lambda)$ at points on the above semicircle when $\beta \to \infty$, it follows that for all sufficiently large values of $\beta \geq \beta_0$, $\Phi_\beta(\lambda)$ will also have no singularities on this semicircle and will satisfy the inequality

$$|\Phi_\beta(\lambda)| < k \qquad (|\lambda| = r, \ \operatorname{Im}\lambda < 0). \tag{2.4.9}$$

Now let $E_\beta^{-1}(\lambda)$ have N_β simple poles in D $(N_\beta \leq N)$. Let us denote these poles by

$$\lambda_i^{(\beta)} \qquad (i = 1, 2, \ldots, N_\beta),$$

and let us define

$$p_\beta(\lambda) = \prod_{i=1}^{N_\beta} (\lambda - \lambda_i^{(\beta)}).$$

Then the function $p_\beta(\lambda)\Phi_\beta(\lambda)$ is regular everywhere in D and on its boundary.

Without restricting the generality of our argument, we may take $r \leq {}^1/_2$, and we have

$$|p_\beta(\lambda)| = \prod_{i=1}^{N_\beta} |\lambda - \lambda_i^{(\beta)}| \leq (2r)^{N_\beta} \leq 1 \qquad (\lambda \in \bar{D}).$$

But then on the basis of (2.4.5), (2.4.9), and the maximum-modulus principle,

$$|p_\beta(\lambda)\Phi_\beta(\lambda)| \leq k \qquad (\lambda \in \bar{D}),$$

i.e.,

$$|\Phi_\beta(\lambda)| \leq k\,|p_\beta(\lambda)|^{-1} \qquad (\lambda \in \bar{D}).$$

Now letting $\beta \to \infty$ and taking into account that $\lambda_i^{(\beta)} \to 0$ and $\Phi_\beta(\lambda) \to \Phi(\lambda)$ everywhere in D, we obtain[9]

$$|\Phi(\lambda)| \leq k\,|\lambda|^{-N} \qquad (\lambda \in \bar{D}). \tag{2.4.10}$$

---

[9] We take the largest possible power of $|\lambda|$ in the denominator of the righthand side of (2. 4. 10) because in the domain D

$$|\lambda| < r \leq \frac{1}{2}.$$

Next, applying the Phragmén-Lindelöf Theorem[10] in connection with (2.4.6), (2.4.8), and (2.4.10), we find that

$$|\Phi(\lambda)| = |2i\lambda E^{-1}(\lambda)G^*(x_0, -\bar{\lambda})| \leq k \qquad (\lambda \in \bar{D}). \quad (2.4.11)$$

Finally, from (2.4.1), it follows that (see the proof of Theorem 1.2.1)

$$G^*(x, -\bar{\lambda}) = xe^{i\lambda x}Z^*(x, -\bar{\lambda}),$$

where

$$\lim_{x \to 0} Z^*(x, -\bar{\lambda}) = I$$

uniformly for $\lambda \in \bar{D}$. Therefore a positive number x_0 can be found such that $|\det Z^*(x_0, -\bar{\lambda})| > {}^1/_2$ and, as a consequence,

$$|\det G^*(x_0, -\lambda)| > \frac{1}{2} x_0^n$$

for all $\lambda \in \bar{D}$. Thus, $[G^*(x_0, -\lambda)]^{-1}$ exists for all $\lambda \in \bar{D}$ and

$$|[G^*(x_0, -\bar{\lambda})]^{-1}| < k_1 \qquad (\lambda \in \bar{D}),$$

where k_1 is some constant.

This, together with (2.4.11), shows that

$$|\lambda E^{-1}(\lambda)| \leq \frac{1}{2}|\Phi(\lambda)|\,|[G^*(x_0, -\bar{\lambda})]^{-1}| \leq \frac{1}{2}kk_1, \qquad (\lambda \in \bar{D}),$$

and the lemma is proved.

---

[10] We first map the domain D onto the upper half-plane in such a way that the point $\lambda = 0$ goes into the point at infinity.

CHAPTER III

The Fundamental Equation

§ 1. Derivation of the Fundamental Equation

1. Assuming, as we did in Chapter II, that the potential $V(x)$ is Hermitian and satisfies the condition (2.1.2), we shall examine the matrix $K(x, y)$ which occurs in the expression (1.3.2) for the special solution $E(x, \lambda)$ of equation (2.1.3). In this section, it will be shown that $K(x, y)$ satisfies a linear integral equation whose kernel can be written explicitly in terms of the scattering matrix $S(\lambda)$, the eigenvalues λ_k^2 of the boundary-value problem (2.1.1), and the matrices M_k that determine the multiplicity of these eigenvalues and the normalization of the corresponding eigenfunctions.

To derive this equation, we begin with (2.3.2) rewritten in the form

$$2i\lambda G(x, \lambda)\,[E^*(-\bar{\lambda})]^{-1} = E(x, -\lambda) - E(x, \lambda)S(-\lambda) \quad (\operatorname{Im}\lambda = 0, \lambda \neq 0),$$

and we replace $E(x, \lambda)$ in this by its expression (1.3.2). As a result

$$2i\lambda G(x, \lambda)\,[E^*(-\bar{\lambda})]^{-1} = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + \int_x^\infty K(x, t)\,[e^{i\lambda t}I - e^{-i\lambda t}S(-\lambda)]\,dt = 2i\sin\lambda x I + e^{-i\lambda x}[I - S(-\lambda)] + \int_x^\infty K(x, t)\,\{e^{i\lambda t}I - e^{-i\lambda t}I + e^{-i\lambda t}[I - S(-\lambda)]\}\,dt,$$

or

$$2i\lambda G(x, \lambda)\,\{[E^*(-\bar{\lambda})]^{-1} - I\} + 2i[\lambda G(x, \lambda) - \sin\lambda x I] = e^{-i\lambda x}[I - S(-\lambda)] + \int_x^\infty K(x, t)e^{i\lambda t}\,dt - \int_x^\infty K(x, t)e^{-i\lambda t}\,dt + \int_x^\infty K(x, t)e^{-i\lambda t}\,dt[I - S(-\lambda)]. \tag{3.1.1}$$

By Theorem 2.3.1, $I-S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$ whose elements can each be represented as the sum of two functions, one which is summable, and the other bounded and square summable over the entire real axis. As to $K(x, t)$, its elements are summable over the interval (x, ∞) for any fixed non-negative value of x in consequence of (1.3.3) and (2.1.2).

Using these facts, we can apply known theorems on the convolution[1] and conclude that the right member of (3.1.1) (and, hence, the left member, as well) is the Fourier transform of

$$F_s(x-t)+K(x, -t)-K(x, t)+\int_x^\infty K(x, u)F_s(u-t)\,du. \quad (3.1.2)$$

We now multiply the left-hand side of (3.1.1) by $e^{-i\lambda y}$ and assuming that $x < y$, we integrate the resulting expression with respect to λ from $-\infty$ to ∞. This integral exists as a principal-valued integral and can be evaluated using contour integration. In fact, the first term in the integrand, namely,

$$2i\lambda G(x, \lambda)\{[E^*(-\bar{\lambda})]^{-1}-I\}e^{-i\lambda y}$$

is regular everywhere in the lower half of the λ-plane except for a finite number of points $\lambda_k = -i\mu_k$ corresponding to the simple poles of $E^{-1}(\lambda)$; moreover, it is continuous for all real values of $\lambda \neq 0$ and bounded in the neighborhood of $\lambda = 0$, $\operatorname{Im} \lambda \leq 0$ (see Chapter II, § 2 and 4).[2] Since by Lemma 1.2.1, $\lambda e^{-i\lambda x}\, G(x, \lambda)$ is bounded in the region $\operatorname{Im} \lambda \leq 0$ for each fixed value of $x \geq 0$, and since by (1.3.2),

$$\lim_{|\lambda|\to\infty} \{[E^*(-\bar{\lambda})]^{-1}-I\} = 0$$

uniformly for $\operatorname{Im} \lambda \leq 0$, Jordan's Lemma may be applied. This yields for $x < y$

$$\lim_{l\to\infty} \int_{-l}^{l} 2i\lambda G(x, \lambda)\{[\mathrm{E}^*(-\bar{\lambda})]^{-1}-I\}e^{-i\lambda y}\,d\lambda =$$

$$= -2\pi i \sum \mathrm{Res} = -4\pi \sum_{k=1}^{p} \lambda_k G(x, \lambda_k)N_k^* e^{-i\lambda_k y}.$$

Here, N_k is the residue of the matrix $E^{-1}(\lambda)$ at the pole λ_k.

---

[1] See, for example, Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford 1948, Theorems 41 and 65.

[2] It is easy to see that the matrix $[E^*(-\bar{\lambda})]^{-1}$ is regular in the halfplane $\operatorname{Im} \lambda < 0$, i.e., where $E^{-1}(\lambda)$ is regular, and has simple poles at the points λ_k. Furthermore, if N_k is the residue of $E^{-1}(\lambda)$ at the pole λ_k, then the residue of $[E^*(-\bar{\lambda})]^{-1}$ at this point is $-N_k^*$.

The second term of the integrand,

$$2i[\lambda G(x,\ \lambda)-\sin \lambda x I],$$

is an entire function of λ and on account of the relation (1.2.8), Jordan's Lemma may also be applied to it, so that we have

$$\lim_{l\to\infty} \int_{-l}^{l} 2i[\lambda G(x,\ \lambda)-\sin \lambda x I]e^{-i\lambda y}\, d\lambda = 0.$$

Thus, we have shown that when $0 \leq x < y$

$$\lim_{l\to\infty} \int_{-l}^{l} \{2i\lambda G(x,\ \lambda)\,[(E^*(-\bar{\lambda}))^{-1}-I]+2i[\lambda G(x,\ \lambda)-$$

$$-\sin \lambda x I]\}e^{-i\lambda y}\, d\lambda = -4\pi \sum_{k=1}^{p} \lambda_k G(x,\ \lambda_k)N_k^* e^{-i\lambda_k y}. \tag{3.1.3}$$

But as noted above, the expression within the braces in the integrand of (3.1.3) is the Fourier transform of (3.1.2) (for fixed x), and therefore the integral in (3.1.3), with $-y = t$, is equal to the matrix (3.1.2) multiplied by 2π. Noting further that $K(x,\ t) = 0$ when $x > t$, we obtain the following result:

$$F_s(x+y)+K(x,\ y)+\int_{x}^{\infty} K(x,\ t)F_s(t+y)\, dt =$$

$$= -2 \sum_{k=1}^{p} \lambda_k G(x,\ \lambda_k)N_k^* e^{-i\lambda_k y}. \qquad (0 \leq x < y) \tag{3.1.4}$$

2. We now convert the right-hand side of (3.1.4) into another form. To this end, we start from (2.2.5) with $\lambda = \lambda_k$:[3]

$$\dot{E}^*(x,\ \lambda_k)E'(x,\ \lambda_k)-\dot{E}^{*\prime}(x,\ \lambda_k)E(x,\ \lambda_k) = 2\lambda_k \int_{x}^{\infty} E^*(t,\ \lambda_k)E(t,\ \lambda_k)\, dt, \tag{3.1.5}$$

and the relations

$$E(\lambda_k)N_k = N_k E(\lambda_k) = 0,$$

$$E(\lambda_k)N_k^{(0)}+\dot{E}(\lambda_k)N_k = N_k^{(0)}E(\lambda_k)+N_k\dot{E}(\lambda_k) = I, \tag{3.1.6}$$

which can be obtained directly from the following expansions:

$$E(\lambda) = E(\lambda_k)+(\lambda-\lambda_k)\dot{E}(\lambda_k)+\ \ldots;\ E^{-1}(\lambda) = (\lambda-\lambda_k)^{-1}N_k+N_k^{(0)}+\ldots$$

---

[3] The dot denotes differentiation with respect to λ.

Let P_k be a Hermitian matrix which is a projection onto the null space[4] of the matrix $E(\lambda_k)$, so that $E(\lambda_k)P_k = 0$. By the use of (3.1.6), it is easy to show that the set of vectors of the form $N_k a$, where a is an arbitrary vector, coincides with the null space of $E(\lambda_k)$. Hence, it follows that the matrices N_k and P_k have the same rank and that

$$P_k N_k = N_k. \tag{3.1.7}$$

We next multiply (3.1.5) on the left by N_k^* and on the right by P_k, and we then let $x \to 0$ in the result. As a consequence, we obtain [see the proof of Theorem 2.2.1 and derivation of (2.2.7)]

$$N_k^* \dot{E}^*(\lambda_k) E'(0, \lambda_k) P_k = 2\lambda_k N_k^* A_k P_k,$$

where

$$A_k = \int_0^\infty E^*(t, \lambda_k) E(t, \lambda_k)\, dt$$

and

$$E'(0, \lambda_k) P_k = \lim_{x \to 0} E'(x, \lambda_k) P_k.$$

One can easily see that A_k is a positive definite[5] Hermitian matrix.

Making use of the second equation in (3.1.6), and then Lemma 1.3.2, we find that

$$\begin{aligned} N_k^* \dot{E}^*(\lambda_k) E'(0, \lambda_k) P_k &= E'(0, \lambda_k) P_k - N_k^{(0)*} E^*(\lambda_k) E'(0, \lambda_k) P_k = \\ &= E'(0, \lambda_k) P_k - N_k^{(0)*} E^{*\prime}(0, \lambda_k) E(\lambda_k) P_k = E'(0, \lambda_k) P_k. \end{aligned}$$

Therefore,

$$E'(0, \lambda_k) P_k = 2\lambda_k N_k^* A_k P_k. \tag{3.1.8}$$

Now,

$$E(x, \lambda_k) P_k = G(x, \lambda_k) E'(0, \lambda_k) P_k,$$

since each side of this equation is a solution of (2.1.3) when $\lambda = \lambda_k$ and satisfies the same initial conditions at $x = 0$. By means of (3.1.8) and (3.1.7), this relation can be put into the form

$$\begin{aligned} E(x, \lambda_k) P_k &= 2\lambda_k G(x, \lambda_k) N_k^* A_k P_k = 2\lambda_k G(x, \lambda_k) N_k^* [P_k A_k P_k + \\ &+ (I - P_k)] = 2\lambda_k G(x, \lambda_k) N_k^* B_k, \end{aligned} \tag{3.1.9}$$

---

[4] The *null space* or *kernel* of a matrix A is the set of vectors a such that $Aa = 0$.

[5] That is, with a corresponding positive definite quadratic form.

where

$$B_k = P_k A_k P_k + (I - P_k)$$

can easily shown to be a positive definite Hermitian matrix.

From the obvious equality,

$$P_k B_k = B_k P_k ,$$

it also follows that

$$P_k B_k^{-1} = B_k^{-1} P_k = P_k B_k^{-1} P_k .$$

This yields

$$P_k B_k^{-\frac{1}{2}} = B_k^{-\frac{1}{2}} P_k = P_k B_k^{-\frac{1}{2}} P_k ,$$

$B_k^{-1/2}$ being a positive definite Hermitian matrix whose square is B_k^{-1}. Hence, we conclude that

$$M_k = P_k B_k^{-\frac{1}{2}} = B_k^{-\frac{1}{2}} P_k$$

is a non-negative Hermitian matrix whose null space coincides with the null space of P_k (therefore, these matrices have the same rank). Furthermore,

$$M_k^2 = P_k B_k^{-1} P_k = P_k B_k^{-1} .$$

M_1, M_2, . . ., and M_p will henceforth be referred to as the *normalization matrices.*[6]

Postmultiplying (3.1.9) by $B_k^{-1/2}$ and then by B_k^{-1}, we obtain the respective results

$$E(x,\ \lambda_k) M_k = 2\lambda_k G(x,\ \lambda_k) N_k^* B_k^{\frac{1}{2}} , \tag{3.1.10}$$

$$E(x,\ \lambda_k) M_k^2 = 2\lambda_k G(x,\ \lambda_k) N_k^* . \tag{3.1.11}$$

---

[6] The term is justified by the fact that the columns of the matrices $E(x,\lambda_k)\ M_k$ yield a complete set of normalized eigenfunctions of the problem (2. 1. 1) corresponding to the eigenvalues λ_k^2.

In fact,

$$M_k \int_0^\infty E^*(x,\ \lambda_k)\ E(x,\ \lambda_k)\ dx M_k = M_k A_k M_k = P_k B_k^{-\frac{1}{2}} P_k A_k P_k B_k^{-\frac{1}{2}} P_k =$$

$$= P_k B_k^{-\frac{1}{2}} [P_k A_k P_k + (I - P_k)] B_k^{-\frac{1}{2}} P_k = P_k .$$

Substituting (3.1.11) in (3.1.4) and then using the expression for $E(x, \lambda_k)$ given by (1.3.2), we finally have $(0 \leq x < y)$

$$F(x+y)+K(x, y)+\int_{x}^{\infty} K(x, t)F(t+y)\,dt = 0. \tag{3.1.12}$$

Here,[7]

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-i\lambda_k t}+F_S(t) =$$

$$= \sum_{k=1}^{p} M_k^2 e^{-i\lambda_k t}+\frac{1}{2\pi}\int_{-\infty}^{\infty} [I-S(\lambda)]e^{i\lambda t}\,d\lambda. \tag{3.1.13}$$

The equation (3.1.12) plays a basic role in the investigation of the problem (2.1.1), and it will be referred to as the *fundamental equation*.

The aggregate of quantities

$$S(\lambda);\quad \lambda_k^2,\ M_k^2 \qquad (k=1, 2, \ldots, p),$$

from which the kernel [i.e., the matrix $F(t)$] of the fundamental equation is constructed, will be called simply the *scattering data* of the problem (2.1.1).

From (3.1.13), it is immediately evident that $F(t)$ is a Hermitian matrix. Its further properties will be examined in the next section.

§ 2. Properties of the Kernel

To begin our study of the properties of the kernel of the fundamental equation, i.e., the matrix $F(t)$ for $t > 0$, we note that (3.1.13) and Theorem 2.3.1 enable us to write

$$F(t) = F^{(1)}(t)+F^{(2)}(t), \tag{3.2.1}$$

where $F^{(1)}(t)$ is summable and $F^{(2)}(t)$ is square summable (and bounded) over the interval $(0, \infty)$.

---

[7] The integral on the right-hand side of (3. 1. 13) is merely a tentative formal expression for the matrix $F_S(t)$ in terms of its Fourier transform $I—S(\lambda)$. However, in the next section, it will be proved that $F_S(t)$ is differentiable for all $t > 0$, and from this it follows that the integral exists as a principal-valued integral:

$$F_S(t) = \lim_{l\to\infty}\frac{1}{2\pi}\int_{-l}^{l} [I-S(\lambda)]e^{i\lambda t}\,d\lambda \qquad (t > 0).$$

With the help of the fundamental equation, we shall now describe the properties of $F(t)$ in more detail.

Let us first show that $F(t)$ is continuous for all positive values of t. In fact, from (3.1.12), we have

$$F(x+y+\Delta y)-F(x+y) = -[K(x,\ y+\Delta y)-K(x,\ y)]-$$
$$-\int_x^{\infty} K(x,\ t)\,[F(t+y+\Delta y)-F(t+y)]\,dt.$$

Since $K(x, y)$ is continuous for $0 < x < y$ (see the remark of Theorem 1.3.1), it suffices to prove that when $0 < x < y$

$$\lim_{\Delta y\to 0}\int_x^{\infty} K(x,\ t)\,[F(t+y+\Delta y)-F(t+y)]\,dt = 0. \qquad (3.2.2)$$

Taking into account (3.2.1) and making use of the estimate

$$|\,K(x,\ t)\,| \le c\sigma\left(\frac{x+t}{2}\right), \qquad c=\frac{1}{2}\,e^{\sigma_1(0)} \qquad (3.2.3)$$

(see (1.3.3)), we find that

$$\left|\int_x^{\infty} K(x,t)\,[F^{(1)}(t+y+\Delta y)-F^{(1)}(t+y)]\,dt\,\right| \le$$
$$\le c\sigma(x)\int_x^{\infty} |\,F^{(1)}(t+y+\Delta y)-F^{(1)}(t+y)\,|\,dt$$

and [8]

$$\left|\int_x^{\infty} K(x,\ t)\,[F^{(2)}(t+y+\Delta y)-F^{(2)}(t+y)]\,dt\,\right|^2 \le$$
$$\le \int_x^{\infty} |\,K(x,\ t)|^2\,dt\cdot\int_x^{\infty} |\,F^{(2)}(t+y+\Delta y)-F^{(2)}(t+y)\,|^2\,dt$$

$(0 < x < y)$. It is clear that the right-hand side of each of these two inequalities tends to zero as $\Delta y \to 0$.

---

[8] We are applying the Schwarz inequality. This is permissible since for $x > 0$

$$\int_x^{\infty} |\,K(x,\ t)\,|^2\,dt \le c^2\sigma(x)\int_x^{\infty} \sigma\left(\frac{x+t}{2}\right)dt = 2c^2\sigma(x)\int_x^{\infty} \sigma(s)\,ds \le$$
$$\le 2c^2\sigma(x)\sigma_1(x) < \infty.$$

This proves the validity of (3.2.2), and hence, the continuity of $F(t)$ for all positive values of t.

It is now not difficult to show that the fundamental equation, which was derived on the assumption that $0 \leq x < y$, also holds for $y = x > 0$ by continuity.

For the further investigation of the properties of $F(t)$, we shall need the following lemma.

Lemma 3.2.1: *Let $f(x)$ and $g(x)$ be non-negative functions in the interval $a \leq x \leq b \leq \infty$, and suppose that $g(x)$ and $f(x)\,g(x)$ are summable in this interval. If*

$$f(x) \leq c + \int_x^b f(t)g(t)\,dt, \qquad (a \leq x \leq b),$$

where c is a positive constant, then

$$f(x) \leq ce^{\int_x^b g(t)\,dt} \qquad (a \leq x \leq b).$$

Proof: From the given inequality, it follows that

$$\frac{f(t)g(t)}{c + \int_t^b f(\tau)g(\tau)\,d\tau} \leq g(t) \qquad (a \leq t \leq b).$$

Hence, integrating this from x to b with $a \leq x \leq b$, we arrive at

$$\log\,[c + \int_x^b f(\tau)g(\tau)\,d\tau] - \log c \leq \int_x^b g(t)\,dt,$$

or

$$f(x) \leq c + \int_x^b f(t)g(t)\,dt \leq ce^{\int_x^b g(t)\,dt}$$

q.e.d.

Starting again with the fundamental equation, we obtain the following inequality: [9]

$$| F(x+y) | \leq | K(x, y) | + \int_x^\infty | F(y+t) | \, | K(x, t) | \,dt \leq$$
$$\leq c\sigma\left(\frac{x+y}{2}\right) + c \int_x^\infty | F(y+t) | \, \sigma\left(\frac{x+t}{2}\right) dt$$

---

[9] The existence of the integral on the right-hand side of this inequality is easily proved using the representation (3.2.1) for $F(t)$ (see the proof of equation (3.2.2)).

(see (3.2.3)). Suppose first that $\sigma(u) \neq 0$ for all positive finite values of u. Then multiplying the preceding inequality by $\sigma^{-1}\{(x+y)/2\}$, we have $(x \leq y)$

$$|F(x+y)|\,\sigma^{-1}\left(\frac{x+y}{2}\right) \leq c+c\sigma^{-1}\left(\frac{x+y}{2}\right)\int_x^\infty |F(t+y)|\,\sigma\left(\frac{x+t}{2}\right)dt \leq$$

$$\leq c+c\sigma^{-1}(y)\int_x^\infty \left[|F(t+y)|\,\sigma^{-1}\left(\frac{t+y}{2}\right)\right]\left[\sigma\left(\frac{t+y}{2}\right)\sigma\left(\frac{t}{2}\right)\right]dt.$$

Now, for each fixed value of $y > 0$, the functions

$$g_y(t) = c\sigma^{-1}(y)\sigma\left(\frac{t+y}{2}\right)\sigma\left(\frac{t}{2}\right)$$

and

$$\left[|F(t+y)|\,\sigma^{-1}\left(\frac{t+y}{2}\right)\right]g_y(t) = c\sigma^{-1}(y)\,|F(t+y)|\,\sigma\left(\frac{t}{2}\right)$$

are easily seen to be summable over the interval (x, ∞) for $x > 0$. Lemma 3.2.1 is therefore applicable to the last inequality, and this yields

$$|F(x+y)|\,\sigma^{-1}\left(\frac{x+y}{2}\right) \leq ce^{\int_x^\infty g_y(t)\,dt} \leq ce^{c\sigma^{-1}(y)\sigma\left(\frac{x+y}{2}\right)\int_x^\infty \sigma\left(\frac{t}{2}\right)dt}.$$

Hence, for $y = x > 0$, it follows that

$$|F(2x)|\,\sigma^{-1}(x) \leq ce^{c\int_x^\infty \sigma\left(\frac{t}{2}\right)dt} \leq c_1.$$

Here, c_1 is a constant since

$$\int_x^\infty \sigma\left(\frac{t}{2}\right)dt = \int_x^\infty dt\int_{\frac{t}{2}}^\infty |V(s)|\,ds = \int_{\frac{x}{2}}^\infty |V(s)|\,ds\int_x^{2s} dt \leq$$

$$\leq 2\int_{\frac{x}{2}}^\infty s\,|V(s)|\,ds = 2\sigma_1\left(\frac{x}{2}\right) \leq 2\sigma_1(0).$$

Thus, it has been shown that if $\sigma(u) \neq 0$ for all finite positive values of u, then

$$|F(t)| \leq c_1 \sigma\left(\frac{t}{2}\right) \qquad (t > 0). \tag{3.2.4}$$

It is easy to see that this estimate holds even if $\sigma(u)$ vanishes. For, suppose that $\sigma(u) = 0$ when $u \geq \alpha > 0$ and $\sigma(u) > 0$ when $2 < u < \alpha$ (this means that $V(x) = 0$ for $x \geq \alpha$). In that case, (3.2.3) shows that $K(x, y) = 0$ for $x+y \geq 2\alpha$, and from the fundamental equation, we then conclude that $F(t) = 0$ for $t \geq 2\alpha$. Formula (3.2.4) is thus true for $t \geq 2\alpha$. In order to show that it holds for $0 < t < 2\alpha$, we again resort to the fundamental equation assuming the values of x and y to be such that $x \leq y$ and $0 < x+y < 2\alpha$. Since $F(t) = 0$ for $t \geq 2\alpha$, the fundamental equation yields for the case in question

$$|F(x+y)| \leq |K(x, y)| + \int_x^{2\alpha-y} |F(t+y)|\,|K(x, t)|\,dt.$$

Hence, using Lemma 3.2.1, we find as in the above argument that

$$|F(x+y)|\,\sigma^{-1}\left(\frac{x+y}{2}\right) \leq ce^{\int_x^{2\alpha-y} g_y(t)\,dt} \leq ce^{\int_x^{\infty} g_y(t)\,dt}$$

for $0 < x+y < 2\alpha$, and this clearly leads to (3.2.4) for $t < 2\alpha$.

Finally, if $\sigma(u) \equiv 0$, then obviously $F(t) \equiv 0$ for positive values of t, so that (3.2.4) is also valid in this case.

Let us next show that $F(t)$ has a continuous derivative $F'(t)$ for all positive t, and that it may be obtained by differentiating the fundamental equation with respect to x, i.e., for $0 < x \leq y$

$$\begin{aligned} F'(x+y) = {} & -K_x(x, y) + K(x, x)F(x+y) - \\ & - \int_x^\infty K_x(x, t)F(t+y)\,dt. \end{aligned} \tag{3.2.5}$$

It clearly suffices to show that the integral in this converges uniformly for x in the range $0 < \varepsilon \leq x$, ε being any given positive number.

By (1.3.9),

$$|K_x(x, t)| \leq \frac{1}{4}\left|V\left(\frac{x+t}{2}\right)\right| + c\sigma\left(\frac{x+t}{2}\right)\sigma(x), \quad c = \frac{1}{2}e^{\sigma_1(0)}. \tag{3.2.6}$$

Using this inequality and (3.2.4), we find that for $0 < \varepsilon \leq x \leq y$ and any $N > x$

$$\int_N^\infty |K_x(x, t)|\,|F(t+y)|\,dt \leq \frac{c_1}{4}\int_N^\infty \left|V\left(\frac{x+t}{2}\right)\right| \sigma\left(\frac{t+y}{2}\right) dt +$$

$$+cc_1\sigma(x)\int_N^\infty \sigma\left(\frac{x+t}{2}\right)\sigma\left(\frac{t+y}{2}\right) dt \leq \frac{c_1}{4}\int_N^\infty \left|V\left(\frac{x+t}{2}\right)\right| \sigma\left(\frac{x+t}{2}\right) dt +$$

$$+cc_1\sigma(\varepsilon)\sigma\left(\frac{N}{2}\right)\int_N^\infty \sigma\left(\frac{t}{2}\right) dt \leq \frac{c_1}{4}\sigma^2\left(\frac{N}{2}\right) + 2cc_1\sigma(\varepsilon)\sigma\left(\frac{N}{2}\right)\sigma_1\left(\frac{N}{2}\right),$$

since

$$\int_N^\infty \left|V\left(\frac{x+t}{2}\right)\right| \sigma\left(\frac{x+t}{2}\right) dt = 2\int_{\frac{N+x}{2}}^\infty \sigma(s)\,|V(s)|\,ds =$$

$$= \sigma^2\left(\frac{N+x}{2}\right) \leq \sigma^2\left(\frac{N}{2}\right)$$

and

$$\int_N^\infty \sigma\left(\frac{t}{2}\right) dt = 2\int_{\frac{N}{2}}^\infty \sigma(s)\,ds = 2\sigma_1\left(\frac{N}{2}\right).$$

The above estimate shows that the integral on the right-hand side of (3.2.5) is uniformly convergent, and so the derivative $F'(t)$, which satisfies this equation, exists. From (3.2.5), it immediately follows that $F'(t)$ is continuous for $t > 0$.

Setting $y = x$ in (3.2.5) and noting that

$$K_x(x, y)\Big|_{x=y} = -\frac{1}{4}V(x) - \frac{1}{2}\int_x^\infty (Vt)K(t, t)\,dt$$

(cf. (1.3.8)), we obtain

$$F'(2x) = \frac{1}{4}V(x) + \frac{1}{2}\int_x^\infty V(t)K(t, t)\,dt + K(x, x)F(2x) -$$

$$-\int_x^\infty K_x(x, t)F(x+t)\,dt.$$

On the basis of this equality, it is now possible to prove that

$$\left| F'(2x) - \frac{1}{4} V(x) \right| \leq c_2 \sigma^2(x), \tag{3.2.7}$$

where c_2 is some constant. For, by (3.2.3), (3.2.4), and (3.2.6), we have

$$\left| \int_x^\infty V(t) K(t, t)\, dt \right| \leq c \int_x^\infty \sigma(t) \, | V(t) | \, dt = \frac{c}{2} \sigma^2(x);$$

$$| K(x, x) F(2x) | \leq c c_1 \sigma^2(x);$$

$$\left| \int_x^\infty K_x(x, t) F(x+t)\, dt \right| \leq \frac{c_1}{4} \int_x^\infty \sigma\left(\frac{x+t}{2}\right) \left| V\left(\frac{x+t}{2}\right) \right| dt +$$

$$+ c c_1 \sigma(x) \int_x^\infty \sigma^2\left(\frac{x+t}{2}\right) dt \leq \frac{c_1}{2} \int_x^\infty \sigma(s) \, | V(s) | \, ds +$$

$$+ 2 c c_1 \sigma^2(x) \int_x^\infty \sigma(s)\, ds \leq \frac{c_1}{4} \sigma^2(x) + 2 c c_1 \sigma^2(x) \sigma_1(x) \leq$$

$$\leq \left[\frac{c_1}{4} + 2 c c_1 \sigma_1(0) \right] \sigma^2(x).$$

We next note that

$$x\sigma(x) = x \int_x^\infty | V(s) | \, ds \leq \int_x^\infty s \, | V(s) | \, ds = \sigma_1(x)$$

and therefore

$$\int_0^\infty x \sigma^2(x)\, dx \leq \int_0^\infty \sigma_1(x) \sigma(x)\, dx \leq \sigma_1(0) \int_0^\infty \sigma(x)\, dx = \sigma_1^2(0).$$

By (3.2.7) and (2.1.2), it follows from this that

$$\int_0^\infty t \, | F'(t) | \, dt < \infty. \tag{3.2.8}$$

We summarize the above results in the following theorem.

THEOREM 3.2.1: *The matrix* $F(t)$ *determined by the scattering data of the non-singular boundary-value problem by means of* (3.1.13) *has* *the following properties;*

(*1*) $$| F(t) | \leq c_1\sigma\left(\frac{t}{2}\right) \qquad (t>0),$$

where c_1 *is a constant;*

(*2*) *The derivative* $F'(t)$ *exists and is continuous for all positive* t, *and*

$$\left| F'(2t) - \frac{1}{4} V(t) \right| \leq c_2\sigma^2(t),$$

where c_2 *is a constant. As a consequence of this,*

$$\int_0^\infty t\,| F'(t) |\,dt < \infty.$$

Corollary: *The following inequalities hold:*

$$\int_0^\infty | F(t) |\,dt < \infty \tag{3.2.9}$$

and

$$\int_0^\infty t\,| F(t) |^2\,dt < \infty. \tag{3.2.10}$$

In fact,

$$| F(t) | \leq \int_t^\infty | F'(s) |\,ds,$$

$$t\,| F(t) | \leq t \int_t^\infty | F'(s) |\,ds \leq \int_t^\infty s\,| F'(s) |\,ds.$$

The inequality (3.2.8) in conjunction with these estimates then leads to

$$\int_0^\infty | F(t) |\,dt \leq \int_0^\infty dt \int_t^\infty | F'(s) |\,ds = \int_0^\infty | F'(s) |\,ds \int_0^s dt =$$

$$= \int_0^\infty s\,| F'(s) |\,ds < \infty,$$

$$\int_0^\infty t\,| F(t) |^2\,dt \leq \int_0^\infty | F(t) |\,dt \int_t^\infty s\,| F'(s) |\,ds \leq \int_0^\infty | F(t) |\,dt \int_0^\infty s\,| F'(s) |\,ds < \infty.$$

REMARK. It is apparent from (3.1.13) that $F'_s(t)$ also exists and satisfies a condition analogous to (3.2.8). This means that estimates similar to (3.2.9) and (3.2.10) will also hold for $F'_s(t)$.

§ 3. Lemmas on Integral Equations with Kernels Dependent on a Sum

Let us denote by $L^p_{(n)}(\varepsilon, \infty)$, $p = 1, 2$, or ∞, the normed linear spaces of vector functions $x(t) = \{x_1(t), \ldots, x_n(t)\}$ in which the norms are defined by[10]

$$||x||_p^{(\varepsilon)} = \left\{ \int_\varepsilon^\infty \sum_{i=1}^n |x_i(t)|^p \, dt \right\}^{\frac{1}{p}} \qquad (p = 1, 2),$$

$$||x||_\infty^{(\varepsilon)} = \max_{1 \le i \le n} \ \operatorname{vrai}\max_{\varepsilon < t < \infty} |x_i(t)| \qquad (p = \infty).$$

If vectors are regarded as row matrices (which we shall do below), then by the definition of the absolute value of a matrix, we can evidently write

$$||x||_1^{(\varepsilon)} = \int_\varepsilon^\infty |x(t)| \, dt.$$

Also, for fixed t, $x(t)$ is a vector in n-dimensional vector space where a scalar product can be introduced in the usual way:

$$(x(t), y(t)) = \sum_{i=1}^n x_i(t)\bar{y}_i(t).$$

By the use of this, it is possible to write $||x||_2^{(\varepsilon)}$ in the form

$$||x||_2^{(\varepsilon)} = \left\{ \int_\varepsilon^\infty (x(t), x(t)) \, dt \right\}^{\frac{1}{2}}.$$

Consider a square matrix $A(t)$ of n-th order whose rows (and hence, columns) are elements of $L^1_{(n)}(\varepsilon, \infty)$, with $\varepsilon \ge 0$. By the theorem on the convolution of two summable functions, the matrix $A(t)$ generates a linear operator $\mathbf{A}$ in $L^1_{(n)}(\varepsilon, \infty)$ defined by

$$\mathbf{A}[x] = \int_\varepsilon^\infty x(\xi) A(t+\xi) \, d\xi, \qquad x(t) \in L^1_{(n)}(\varepsilon, \infty). \tag{3.3.1}$$

---

[10] The vrai maximum or "true" maximum of $|x_i(t)|$ is the smallest value of M for which $|x_i(t)| \le M$ almost everywhere for $\varepsilon < t < \infty$.

Let us find an estimate for the norm of the operator **A**. We have

$$\| \mathbf{A}[x] \|_1^{(\varepsilon)} = \int_{\varepsilon}^{\infty} | \mathbf{A}[x] | \, dt \leq \int_{\varepsilon}^{\infty} dt \int_{\varepsilon}^{\infty} | x(\xi) | \; | A(t+\xi) | \, d\xi =$$

$$= \int_{\varepsilon}^{\infty} | x(\xi) | \, d\xi \int_{\varepsilon+\xi}^{\infty} | A(s) | \, ds \leq \| x \|_1^{(\varepsilon)} \int_{\varepsilon}^{\infty} | A(s) | \, ds$$

and therefore

$$\| \mathbf{A} \| \leq \int_{\varepsilon}^{\infty} | A(s) | \, ds.$$

LEMMA 3.3.1: **A** *is a completely continuous operator.*

Proof: Let $\| x \|_1^{(\varepsilon)} = 1$ be the unit sphere in the space $L^1_{(n)}(\varepsilon, \infty)$, and denote by **R** the set of vectors of the form

$$y(t) = \mathbf{A}[x], \qquad \| x \|_1^{(\varepsilon)} \leq 1.$$

From this definition, we immediately have

$$(\alpha) \quad \| y \|_1^{(\varepsilon)} \leq \| \mathbf{A} \| \cdot \| x \|_1^{(\varepsilon)} \leq \| \mathbf{A} \|, \qquad y(t) \in \mathbf{R}.$$

Using (3.3.1), we further have that

$$\int_{\varepsilon}^{\infty} | y(t+h) - y(t) | \, dt \leq \int_{\varepsilon}^{\infty} dt \int_{\varepsilon}^{\infty} | x(\xi) | \; | A(t+h+\xi) - A(t+\xi) | \, d\xi =$$

$$= \int_{\varepsilon}^{\infty} | x(\xi) | \, d\xi \int_{\varepsilon+\xi}^{\infty} | A(s+h) - A(s) | \, ds \leq \| x \|_1^{(\varepsilon)} \cdot \int_{\varepsilon}^{\infty} | A(s+h) - A(s) | \, ds.$$

Hence, for any $\delta > 0$,

$$(\beta) \qquad \int_{\varepsilon}^{\infty} | y(t+h) - y(t) | \, dt < \delta, \qquad y(t) \in \mathbf{R},$$

for all values of h sufficiently small.

By applying (3.3.1) again, we finally have

$$\int_{N}^{\infty} | y(t) | \, dt \leq \int_{N}^{\infty} dt \int_{\varepsilon}^{\infty} | x(\xi) | \; | A(t+\xi) | \, d\xi =$$

$$= \int_{\varepsilon}^{\infty} | x(\xi) | \, d\xi \int_{N+\xi}^{\infty} | A(s) | \, ds \leq \| x \|_1^{(\varepsilon)} \int_{N}^{\infty} | A(s) | \, ds$$

and hence for any $\delta > 0$,

$$(\gamma) \qquad \int_{N}^{\infty} | y(t) | \, dt < \delta, \qquad y(t) \in \mathbf{R},$$

for N sufficiently large.

From (α), (β), and (γ), it follows that the set **R** is compact in $L^1_{(n)}(\varepsilon, \infty)$,[11] and with this the complete continuity of **A** is proved.

COROLLARY: *The equation*

$$a(t)+\lambda x(t)+\mathbf{A}[x] = 0,$$

where $a(t) \in L^1_{(n)}(\varepsilon, \infty)$, *has a unique solution* $x(t)$ *if the only solution of the homogeneous equation*

$$\lambda x(t)+\mathbf{A}[x] = 0$$

in $L^1_{(n)}(\varepsilon, \infty)$ *is the trivial solution.*

Below, we shall also deal with functions which do not belong to the spaces introduced above, but are instead expressible as the sum of two functions, one belonging to $L^1_{(n)}(\varepsilon, \infty)$, and the other to both $L^2_{(n)}(\varepsilon, \infty)$ and $L^\infty_{(n)}(\varepsilon, \infty)$. The space of all such functions will be denoted by $\hat{L}_{(n)}(\varepsilon, \infty)$.

LEMMA 3.3.2: *If the rows of a matrix* $B(t)$ *belong to* $\hat{L}_{(n)}(\varepsilon, \infty)$, $\varepsilon \geq 0$, *hen any solution of*

$$f(t)+\int_{\varepsilon}^{\infty} f(\xi)B(t+\xi)\,d\xi = 0$$

in $\hat{L}_{(n)}(\varepsilon, \infty)$ *is an element of* $L^\infty_{(n)}(\varepsilon, \infty)$ *and hence an element of* $L^2_{(n)}(\varepsilon, \infty)$.

Proof: Let $f(t)$ be any solution of the equation, and suppose that

$$f(t) = f^{(1)}(t)+f^{(2)}(t), \qquad B(t) = B^{(1)}(t)+B^{(2)}(t),$$

where $f^{(1)}(t)$ and the rows of $B^{(1)}(t)$ belong to $L^1_{(n)}(\varepsilon, \infty)$, and $f^{(2)}(t)$ and the rows of $B^{(2)}(t)$ belong to both $L^2_{(n)}(\varepsilon, \infty)$ and $L^\infty_{(n)}(\varepsilon, \infty)$. Then

$$f^{(1)}(t)+\int_{\varepsilon}^{\infty} f^{(1)}(\xi)B^{(1)}(t+\xi)\,d\xi = -\left\{f^{(2)}(t)+\int_{\varepsilon}^{\infty} f^{(1)}(\xi)B^{(2)}(t+\xi)\,d\xi+\right.$$
$$\left.+\int_{\varepsilon}^{\infty} f^{(2)}(\xi)B^{(1)}(t+\xi)\,d\xi+\int_{\varepsilon}^{\infty} f^{(2)}(\xi)B^{(2)}(t+\xi)\,d\xi\right\}.$$

We can now obviously find a matrix $B^{(3)}(t)$ with rows in $L^\infty_{(n)}(\varepsilon, \infty)$ such that the matrix

$$C(t) = B^{(1)}(t)-B^{(3)}(t)$$

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[11] See, for example, V.I. Smirnov, *Course of Higher Mathematics* Vol. 5, Chap. V., § 91 (in Russian).

satisfies the inequality

$$\int_{\varepsilon}^{\infty} |C(t)|\,dt < 1. \tag{3.3.2}$$

Hence,

$$f^{(1)}(t) + \int_{\varepsilon}^{\infty} f^{(1)}(\xi)C(t+\xi)\,d\xi = g(t), \tag{3.3.3}$$

where

$$g(t) = -f^{(2)}(t) - \int_{\varepsilon}^{\infty} f^{(1)}(\xi)B^{(2)}(t+\xi)\,d\xi - \int_{\varepsilon}^{\infty} f^{(2)}(\xi)B^{(1)}(t+\xi)\,d\xi - \\ - \int_{\varepsilon}^{\infty} f^{(2)}(\xi)B^{(2)}(t+\xi)\,d\xi - \int_{\varepsilon}^{\infty} f^{(1)}(\xi)B^{(3)}(t+\xi)\,d\xi.$$

It is easy to see that $g(t)$ belongs to $L^{\infty}_{(n)}(\varepsilon, \infty)$.

Because of (3.3.2), the method of successive approximations may be used to show that (3.3.3) has a unique solution, the approximations converging to $f^{(1)}(t)$ in the metric of $L^{\infty}_{(n)}(\varepsilon, \infty)$. Therefore, $f^{1}_{(n)}(t) \in L^{\infty}_{(n)}(\varepsilon, \infty)$, and since by assumption, $f^{(1)}(t)$ is in $L^{1}_{(n)}(\varepsilon, \infty)$, it follows that $f^{(1)}(t) \in L^{2}_{(n)}(\varepsilon, \infty)$. But then $f(t) = f^{(1)}(t) + f^{(2)}(t)$ belongs to both $L^{\infty}_{(n)}(\varepsilon, \infty)$ and $L^{2}_{(n)}(\varepsilon, \infty)$.

COROLLARY. *If the rows of $A(t)$ belong to $L^{1}_{(n)}(\varepsilon, \infty)$, then any solution of*

$$x(t) + \int_{\varepsilon}^{\infty} x(\xi)A(t+\xi)\,d\xi = 0 \tag{3.3.4}$$

in $L^{1}_{(n)}(\varepsilon, \infty)$ belongs to both $L^{\infty}_{(n)}(\varepsilon, \infty)$ and $L^{2}_{(n)}(\varepsilon, \infty)$.

REMARK. In a similar way, it may be shown that if the rows of $A(t)$ belong to $L^{1}_{(n)}(\varepsilon, \infty)$, $\varepsilon \leq 0$, then any solution of (3.3.4) in $L^{\infty}_{(n)}(\varepsilon, \infty)$ belongs to both $L^{1}_{(n)}(\varepsilon, \infty)$ and $L^{2}_{(n)}(\varepsilon, \infty)$.

LEMMA 3.3.3: *Let the rows of a matrix $\Phi(t)$ belong to $\hat{L}_{(n)}(-\infty, \infty)$ and suppose that its Fourier transform*

$$\tilde{\Phi}(\lambda) = \int_{-\infty}^{\infty} \Phi(t)e^{-i\lambda t}\,dt$$

is continuous over the entire real axis and vanishes when $|\lambda| \to \infty$.

Then for any $\varepsilon > -\infty$, *the operator*

$$\Phi[x] = \int_{\varepsilon}^{\infty} x(\xi)\Phi(t+\xi)\,d\xi$$

is completely continuous in $L^2_{(n)}(\varepsilon, \infty)$.

Proof: The operator Φ is defined everywhere in $L^2_{(n)}(\varepsilon, \infty)$ as a consequence of known theorems on the convolution.[12]

Because of the properties of $\widetilde{\Phi}(\lambda)$, it is possible to construct a sequence of bounded and continuously differentiable matrix functions $\widetilde{\Phi}_k(\lambda)$ such that

$$\max_{-\infty\leq\lambda\leq\infty} |\widetilde{\Phi}_k(\lambda)-\widetilde{\Phi}(\lambda)| < \frac{1}{k} \qquad (k = 1, 2, \ldots). \quad (3.3.5)$$

If $\Phi_k(t)$ is the matrix having $\widetilde{\Phi}_k(\lambda)$ as its Fourier transform, then[13]

$$\int_{-\infty}^{\infty} |\Phi_k(t)|^2\,dt<\infty, \qquad \int_{-\infty}^{\infty} t^2\,|\Phi_k(t)|^2\,dt<\infty,$$

and therefore

$$\int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty} |\Phi_k(t+\xi)|^2\,dt\,d\xi = \int_{\varepsilon}^{\infty} dt \int_{\varepsilon+t}^{\infty} |\Phi_k(s)|^2\,ds =$$

$$= \int_{2\varepsilon}^{\infty} |\Phi_k(s)|^2\,ds \int_{\varepsilon}^{s-\varepsilon} dt = \int_{2\varepsilon}^{\infty} (s-2\varepsilon)\,|\Phi_k(s)|^2\,ds < \infty.$$

Hence, the operators

$$\Phi_k[x]= \int_{\varepsilon}^{\infty} x(\xi)\Phi_k(t+\xi)\,d\xi \qquad (k=1, 2, \ldots)$$

are completely continuous in $L^2_{(n)}(\varepsilon, \infty)$.

Now, since

$$\Phi_k[x] = \int_{\varepsilon}^{\infty} x(\xi)\Phi_k(t+\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{x}(-\lambda)\widetilde{\Phi}_k(\lambda)e^{i\lambda t}\,d\lambda,$$

$$\Phi[x] = \int_{\varepsilon}^{\infty} x(\xi)\Phi(t+\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{x}(-\lambda)\widetilde{\Phi}(\lambda)e^{i\lambda t}\,d\lambda,$$

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[12] Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 1948, p. 90, Theorems 64 and 65.

[13] Ibid, p. 92, Theorem 68.

where

$$\tilde{x}(\lambda) = \int_{\varepsilon}^{\infty} x(t) e^{-i\lambda t}\, dt,$$

this yields

$$\{ \| \Phi_k[x] - \Phi[x] \|_2^{(\varepsilon)} \}^2 = \int_{\varepsilon}^{\infty} \big((\Phi_k - \Phi)[x],\ (\Phi_k - \Phi)[x] \big)\, dt =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \big(\tilde{x}(-\lambda)[\tilde{\Phi}_k(\lambda) - \tilde{\Phi}(\lambda)],\ \tilde{x}(-\lambda)[\tilde{\Phi}_k(\lambda) - \tilde{\Phi}(\lambda)] \big)\, d\lambda. \tag{3.3.6}$$

But for any vector $a = (a_1, \ldots, a_n)$ and matrix $A = \| \alpha_{jk} \|$,[14]

$$(aA,\ aA) \leq n^2 \, |A|^2 \, (a,\ a).$$

Therefore, from (3.3.5) and (3.3.6), we conclude that

$$\{ \| \Phi_k[x] - \Phi[x] \|_2^{(\varepsilon)} \}^2 \leq \frac{n^2}{2\pi k^2} \int_{-\infty}^{\infty} \big(\tilde{x}(-\lambda),\ \tilde{x}(-\lambda) \big)\, d\lambda =$$

$$= \frac{n^2}{k^2} \int_{\varepsilon}^{\infty} \big(x(t),\ x(t) \big)\, dt = \left\{ \frac{n}{k} \| x \|_2^{(\varepsilon)} \right\}^2,$$

and hence

$$\| \Phi_k - \Phi \| \leq \frac{n}{k}.$$

This shows that Φ_k converges uniformly to Φ when $k \to \infty$ and this implies that the operator Φ is also completely continuous in $L^2_{(n)}(\varepsilon, \infty)$.

[14] In fact, denoting $aA = b = (b_1, \ldots, b_n)$, we have

$$|b_k|^2 = \Big| \sum_{j=1}^{n} a_j \alpha_{jk} \Big|^2 \leq \sum_{j=1}^{n} |a_j|^2 \cdot \sum_{j=1}^{n} |\alpha_{jk}|^2 \leq (a,\ a) \cdot n |A|^2$$

and therefore,

$$(b,\ b) = \sum_{k=1}^{n} |b_k|^2 \leq n^2 \, |A|^2 \, (a,\ a).$$

COROLLARY: *If* $\varepsilon \geq 0$ *and the rows of the matrix* $\Phi(t)$ *belong to* $L^1_{(n)}(\varepsilon, \infty)$, *then the operator*

$$\boldsymbol{\Phi}[x] = \int_{\varepsilon}^{\infty} x(\xi)\Phi(t+\xi)\,d\xi$$

is completely continuous in $L^2_{(n)}(\varepsilon, \infty)$.

For, only the values of $\Phi(t)$ for $t \geq 2\varepsilon$ enter into the expression for $\boldsymbol{\Phi}[x]$. Thus, without changing the definition of the operator $\boldsymbol{\Phi}$, we may assume that $\Phi(t) = 0$ for $t < \varepsilon$, and then, it is easy to see that the conditions of Lemma 3.3.3 are fulfilled.

§ 4. Existence of Solutions

Let $F_s(t)$ be a square matrix of n-th order satisfying the following conditions:

1°. its rows belong to $\hat{L}_{(n)}(-\infty, \infty)$;
2°. its Fourier transform is

$$\int_{-\infty}^{\infty} F_s(t)e^{-i\lambda t}\,dt = I - S(\lambda),$$

where $S(\lambda)$ is a unitary matrix such that

$$S(-\lambda) = S^*(\lambda).$$

It is easy to see that this property of $S(\lambda)$ is equivalent to $F_s(t)$ being a Hermitian matrix.[15]

We recall that if $S(\lambda)$ is the scattering matrix of the non-singular boundary-value problem (2.1.1), then $I—S(\lambda)$ is the Fourier transform of a matrix $F_s(t)$ having the properties 1° and 2° (see § 3, Chapter II).

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[15] In fact, if $S(—\lambda) = S^*(\lambda)$, then

$$\int_{-\infty}^{\infty} F_s^*(t)e^{-i\lambda t}\,dt = [\int_{-\infty}^{\infty} F_s(t)e^{i\lambda t}\,dt]^* = [I—S(—\lambda)]^* = I—S(\lambda) = \int_{-\infty}^{\infty} F_s(t)e^{-i\lambda t}\,dt$$

and this implies that $F_s^*(t) = F_s(t)$.

Conversely, if $F_s^*(t) = F_s(t)$, then

$$I—S^*(\lambda) = \int_{-\infty}^{\infty} F_s^*(t)e^{i\lambda t}\,dt = \int_{-\infty}^{\infty} F_s(t)e^{i\lambda t}\,dt = I—S(—\lambda)$$

and hence $S(—\lambda) = S^*(\lambda)$.

Let the vector $x(t)$ belong to $L^2_{(n)}(-\infty, \infty)$, and let $\tilde{x}(\lambda)$ be its Fourier transform:

$$\tilde{x}(\lambda) = \int_{-\infty}^{\infty} x(t)e^{-i\lambda t}\,dt.$$

From theorems on the convolution,[16] it follows that

$$y(t) = \int_{-\infty}^{\infty} x(\xi)F_s(t+\xi)\,d\xi$$

exists for almost all t and belongs to $L^2_{(n)}(-\infty, \infty)$; moreover, its Fourier transform is $\tilde{x}(-\lambda)[I-S(\lambda)]$. Therefore

$$y(t) = \int_{-\infty}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = x(-t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(-\lambda)S(\lambda)e^{i\lambda t}\,d\lambda. \tag{3.4.1}$$

Now choose some $\varepsilon \geq 0$ and consider the linear operator

$$\mathbf{F}_s[x] = \int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi$$

defined in $L^2_{(n)}(\varepsilon, \infty)$. Supposing that $x(t) = 0$ for $t < \varepsilon$, we find from (3.4.1) that

$$\int_{\varepsilon}^{\infty} (\mathbf{F}_s[x], \ \mathbf{F}_s[x])\,dt \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} (\tilde{x}(-\lambda)S(\lambda), \ \tilde{x}(-\lambda)S(\lambda))\,d\lambda$$

for any $x(t) \in L^2_{(n)}(\varepsilon, \infty)$ [which, of course, is zero outside the interval (ε, ∞)]. From the fact that $S(\lambda)$ is unitary, we then get

$$\int_{\varepsilon}^{\infty} (\mathbf{F}_s[x], \ \mathbf{F}_s[x])\,dt \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} (\tilde{x}(-\lambda), \ \tilde{x}(-\lambda))\,d\lambda = \int_{\varepsilon}^{\infty} (x(t), x(t))\,dt.$$

Therefore, for $|\nu| > 1$, the only solution of

$$\nu x(t) + \int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)d\xi = 0$$

in $L^2_{(n)}(\varepsilon, \infty)$ is the trivial solution.

---

[16] See footnote 12 on p. 74.

Again, from (3.4.1), it follows that for any $x(t) \in L^2_{(n)}(\varepsilon, \infty)$ with $x(t) = 0$ for $t < \varepsilon$,

$$\int_{\varepsilon}^{\infty} (\mathbf{F}_s[x], x(t))\, dt = -\frac{1}{2\pi} \int_{-\infty}^{\infty} (\tilde{x}(-\lambda)S(\lambda), \tilde{x}(\lambda))\, d\lambda. \qquad (3.4.2)$$

Using the unitariness of $S(\lambda)$ and the Schwarz inequality, we see that

$$|(\tilde{x}(-\lambda)S(\lambda), \tilde{x}(\lambda))| \leq \frac{1}{2}[(\tilde{x}(-\lambda), \tilde{x}(-\lambda)) + (\tilde{x}(\lambda), \tilde{x}(\lambda))]$$

and therefore

$$\left|\frac{1}{2\pi} \int_{-\infty}^{\infty} (\tilde{x}(-\lambda)S(\lambda), \tilde{x}(\lambda))\, d\lambda\right| \leq \int_{\varepsilon}^{\infty} (x(t), x(t))\, dt.$$

This inequality in conjunction with (3.4.2) shows that

$$\mathrm{Re}\, \{ \int_{\varepsilon}^{\infty} (x(t), x(t))\, dt \pm \int_{\varepsilon}^{\infty} (\mathbf{F}_s[x], x(t))\, dt\} \geq 0,$$

in which equality is possible if and only if

$$(\tilde{x}(-\lambda)S(\lambda), \tilde{x}(\lambda)) = \pm\frac{1}{2}[(x(-\lambda), x(-\lambda)) + (\tilde{x}(\lambda), x(\lambda))]. \qquad (3.4.3)$$

We now prove that (3.4.3) is equivalent to

$$x(-\lambda)S(\lambda) = \pm\tilde{x}(\lambda). \qquad (3.4.4)$$

In fact, from (3.4.4), it follows that

$$(\tilde{x}(-\lambda)S(\lambda), \tilde{x}(\lambda)) = \pm({}^{x}(\lambda), x(\lambda))$$

and because $S(\lambda)$ is unitary,

$$(\tilde{x}(-\lambda), x(-\lambda)) = (x(-\lambda)S(\lambda), x(-\lambda)S(\lambda)) = (x(\lambda), \tilde{x}(\lambda)).$$

Hence, we obtain (3.4.3).

Now, suppose that (3.4.3) holds, and set

$$\tilde{x}(-\lambda)S(\lambda) = \pm\tilde{x}(\lambda) + z(\lambda).$$

Again using the fact that $S(\lambda)$ is unitary, we conclude that

$$(\tilde{x}(-\lambda), \tilde{x}(-\lambda)) = (\tilde{x}(\lambda), \tilde{x}(\lambda)) \pm (\tilde{x}(\lambda), z(\lambda)) \pm$$
$$\pm (z(\lambda), \tilde{x}(\lambda)) + (z(\lambda), z(\lambda)),$$

and by (3.4.3), we have

$$\pm(\tilde{x}(\lambda),\ \tilde{x}(\lambda))+(z(\lambda),\ \tilde{x}(\lambda)) = \pm\frac{1}{2}[(\tilde{x}(-\lambda),\ \tilde{x}(-\lambda))+$$
$$+(\tilde{x}(\lambda),\ \tilde{x}(\lambda))].$$

The last two equations show that $\big(z(\lambda), \tilde{x}(\lambda)\big)$ is real, and so

$$(z(\lambda),\ z(\lambda)) = (\tilde{x}(-\lambda),\ \tilde{x}(-\lambda))-(\tilde{x}(\lambda),\ \tilde{x}(\lambda))\mp 2(z(\lambda),\ \tilde{x}(\lambda)),$$
$$(z(\lambda\ ,\ \tilde{x}(\lambda)) = \pm[(\tilde{x}(-\lambda),\ \tilde{x}(-\lambda))-(\tilde{x}(\lambda),\ \tilde{x}(\lambda))],$$

i.e., $\big(z(\lambda), z(\lambda)\big) = 0$, and therefore $z(\lambda) = 0$.

From (3.4.1) and the uniqueness of the Fourier transform, it follows also that (3.4.4) is equivalent to

$$\pm x(t)+\int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty$$

for the case under consideration $\big(x(t) = 0$ for $t < \varepsilon\big)$.

Suppose now that $x(t)$ is a vector in $L^2_{(n)}(\varepsilon, \infty)$ (identically zero for $t < \varepsilon$) which is a solution of

$$\pm x(t)+\int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty.$$

Then using the relation

$$\tilde{x}(-\lambda)S(\lambda) = \pm\tilde{x}(\lambda)$$

proved above and (3.4.1), we obtain the result

$$\pm x(t)+\int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = \pm x(t)+x(-t)\mp x(t),$$

or

$$\pm x(t)+\int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = x(-t), \qquad -\infty < t < \infty.$$

Thus, we have proved the following lemma.

LEMMA 3.4.1: *Let* $\varepsilon \geq 0$ *and suppose that* $F_s(t)$ *satisfies the conditions* 1° *and* 2°. *Then*

(1) for $|\nu| > 1$, *the equation*

$$\nu x(t)+\int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = 0$$

has no non-trivial solution in $L^2_{(n)}(\varepsilon, \infty)$;

(2) *for any* $x(t) \in L^2_{(n)}(\varepsilon, \infty)$,

$$\operatorname{Re}\left\{\int_\varepsilon^\infty (x(t),\, x(t))\, dt \pm \int_\varepsilon^\infty \Big(\int_\varepsilon^\infty x(\xi)F_s(t+\xi)\, d\xi,\ x(t)\Big)\, dt\right\} \geq 0,$$

quality being achieved if and only if

$$\tilde{x}(-\lambda)S(\lambda) = \pm\tilde{x}(\lambda) \qquad \Big(\tilde{x}(\lambda) = \int_\varepsilon^\infty x(t)e^{-i\lambda t}\, dt\Big).$$

This relation is equivalent to

$$\pm x(t) + \int_\varepsilon^\infty x(\xi)F_s(t+\xi)\, d\xi = 0, \qquad \varepsilon \leq t < \infty,$$

(3) *if* $x(t)$ *belongs to* $L^2_{(n)}(\varepsilon, \infty)$ $[x(t) \equiv 0$ *for* $t < \varepsilon]$ *and satisfies the equation*

$$\pm x(t) + \int_\varepsilon^\infty x(\xi)F_s(t+\xi)\, d\xi = 0, \qquad \varepsilon \leq t < \infty,$$

then

$$\pm x(t) + \int_\varepsilon^\infty x(\xi)F_s(t+\xi)\, d\xi = x(-t), \qquad -\infty < t < \infty.$$

COROLLARY: *Let*

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-\mu_k t} + F_s(t),$$

where all M_k *are Hermitian matrices and* $\mu_k > 0$. *Then, for* $\varepsilon \geq 0$ *a vector* $x(t)$ *in* $L^2_{(n)}(\varepsilon, \infty)$ *is a solution of*

$$x(t) + \int_\varepsilon^\infty x(\xi)F(t+\xi)\, d\xi = 0, \qquad \varepsilon \leq t < \infty \tag{3.4.5}$$

if and only if its Fourier transform

$$\tilde{x}(\lambda) = \int_\varepsilon^\infty x(t)e^{-i\lambda t}\, dt$$

satisfies the conditions

$$\left.\begin{aligned} &\tilde{x}(-i\mu_k)M_k = 0 \qquad &&(k = 1, 2, \dots, p), \\ &\tilde{x}(-\lambda)S(\lambda) = \tilde{x}(\lambda) \qquad &&(-\infty < \lambda < \infty). \end{aligned}\right\} \tag{3.4.6}$$

Proof: If $x(t) \in L^2_{(n)}(\varepsilon, \infty)$ and satisfies (3.4.5), then by forming the scalar product of both sides of this equation with $x(t)$ and integrating, we obtain

$$\int_\varepsilon^\infty (x(t), x(t))\,dt + \int_\varepsilon^\infty \Big(\int_\varepsilon^\infty x(\xi) F_s(t+\xi)\,d\xi,\ x(t)\Big)\,dt + \sum_{k=1}^{p} (\tilde{x}(-i\mu_k) M_k^2,\ \tilde{x}(-i\mu_k)) = 0,$$

or

$$\int_\varepsilon^\infty (x(t), x(t))\,dt + \int_\varepsilon^\infty \Big(\int_\varepsilon^\infty x(\xi) F_s(t+\xi)\,d\xi,\ x(t)\Big)\,dt + \sum_{k=1}^{p} (\tilde{x}(-i\mu_k) M_k,\ \tilde{x}(-i\mu_k) M_k) = 0.$$

Taking the real part of both sides of this equation and making use of statement (2) of Lemma 3.4.1 we get the result (3.4.6).

The sufficiency of (3.4.6) can be verified directly.

LEMMA 3.4.2: *If* $\varepsilon > 0$ *and the rows of* $F_s(t)$ *satisfy conditions* 1° *and* 2° *and belong to* $L^1_{(n)}(\varepsilon, \infty)$, *then for* $|\nu| \geqq 1$ *the only solution of*

$$\nu x(t) + \int_\varepsilon^\infty x(\xi) F_s(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty$$

in $L^2_{(n)}(\varepsilon, \infty)$ *is the trivial solution.*

Proof: The validity of this assertion for $|\nu| > 1$ was proved in Lemma 3.4.1. Suppose $x(t) \in L^2_{(n)}(\varepsilon, \infty)$ and is a solution of

$$\pm x(t) + \int_\varepsilon^\infty x(\xi) F_s(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty,$$

Assuming $x(t) = 0$ for $t < \varepsilon$, we have by statement (3) of Lemma 3.4.1 that

$$\pm x(t) + \int_\varepsilon^\infty x(\xi) F_s(t+\xi)\,d\xi = x(-t), \qquad -\infty < t < \infty.$$

Hence, for any value of h,

$$\pm x(t+h) + \int_\varepsilon^\infty x(\xi) F_s(t+h+\xi)\,d\xi = x(-t-h),$$

$$\pm x(t-h) + \int_\varepsilon^\infty x(\xi) F_s(t-h+\xi)\,d\xi = x(-t+h).$$

Choosing $0 < h < \varepsilon/2$ and recalling that $x(t) = 0$ for $t < \varepsilon$, we conclude from the last two equations that

$$\pm[x(t+h)+x(t-h)]+\int_{\varepsilon-h}^{\infty}[x(\eta+h)+x(\eta-h)]F_s(t+\eta)\,d\eta =$$
$$= x(-t+h)+x(-t-h).$$

Hence

$$\pm z_h(t)+\int_{\frac{\varepsilon}{2}}^{\infty} z_h(\eta)F_s(t+\eta)\,d\eta = 0, \qquad \frac{\varepsilon}{2} \leq t < \infty, \qquad (3.4.7)$$

where

$$z_h(t) = x(t+h)+x(t-h), \qquad 0 < h < \frac{\varepsilon}{2}. \qquad (3.4.8)$$

Now, if $x(t)$ were not equal to zero almost everywhere in the interval (ε, ∞), equation (3.4.7) would then have an infinite number of linearly independent solutions (3.4.8) in $L^2_{(n)}(\varepsilon/2, \infty)$. But this is impossible since under our assumptions, the rows of $F_s(t)$ clearly belong to $L^1_{(n)}(\alpha, \infty)$ for any $\alpha > -\infty$ and according to the corollary to Lemma 3.3.3, the operator

$$\int_{\frac{\varepsilon}{2}}^{\infty} x(\xi)F_s(t+\xi)\,d\xi$$

s completely continuous in $L^2_{(n)}(\varepsilon/2, \infty)$.

Theorem 3.4.1: *Let* $\varepsilon > 0$ *and suppose that the rows of* $F_s(t)$ *satisfy conditions* 1° *and* 2° *and belong to* $L^1_{(n)}(\varepsilon, \infty)$. *If*

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-\mu_k t}+F_s(t),$$

where all M_k *are Hermitian matrices and all* $\mu_k > 0$,
then for any $f(t) \in L^1_{(n)}(\varepsilon, \infty)$, *the equation*

$$f(t)+x(t)+\int_{\varepsilon}^{\infty} x(\xi)F(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty$$

has a unique solution in $L^1_{(n)}(\varepsilon, \infty)$.

Proof: According to the corollaries to Lemmas 3.3.1 and 3.3.2, it suffices to prove that the homogeneous equation

$$x(t)+\int_{\varepsilon}^{\infty} x(\xi)F(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty$$

has no non-trivial solution in $L^2_{(n)}(\varepsilon, \infty)$.

But by the corollary to Lemma 3.4.1, every solution $x(t)$ of this equation belonging to $L^2_{(n)}(\varepsilon, \infty)$ also satisfies

$$x(t)+\int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty.$$

However, by Lemma 3.4.2, the only solution of this equation belonging to $L^2_{(n)}(\varepsilon, \infty)$ is the trivial solution.

THEOREM 3.4.2: *Suppose that the rows of $F_s(t)$ satisfy conditions 1° and 2° and belong to $L^1_{(n)}(\varepsilon, \infty)$, with $\varepsilon > 0$, and that $f(t)$ belongs to $L^2_{(n)}(\varepsilon, \infty)$. Then a solution to the equation*

$$f(t)+x(t)+\int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad \varepsilon \leq t < \infty,$$

exists (by the method of successive approximations).

Proof: By the corollary to Lemma 3.3.3, the operator

$$\mathbf{F}_s[x] = \int_{\varepsilon}^{\infty} x(\xi)F_s(t+\xi)\,d\xi$$

s completely continuous in $L^2_{(n)}(\varepsilon, \infty)$. Moreover, the fact that $F_s(t)$ is Hermitian implies that $\mathbf{F}_s$ is a self-adjoint operator, and by Lemma 3.4.2, the modulus of each of its eigenvalues is less than one. But then $\|\mathbf{F}_s\| < 1$ and as a consequence, the Neumann series

$$(\mathbf{I}+\mathbf{F}_s)^{-1} = \mathbf{I}-\mathbf{F}_s+\mathbf{F}_s^2- \dots,$$

where $\mathbf{I}$ is the identity operator, converges, q.e.d.

From Theorems 3.4.1 and 3.4.2, it follows in particular that the fundamental equation (3.1.12) [in which $F(t)$ is determined by the scattering data of the non-singular boundary-value problem (2.1.1)] has a unique solution for any $x > 0$, and in the event that the problem has no point spectrum, it is possible to solve the fundamental equation by the method of successive approximations-

However, Theorem 3.4.1 is not true in general when $\varepsilon = 0$ and therefore yields no information concerning the existence of a solution of the fundamental equation for $x = 0$. This question will be dealt with in the next section.

§ 5. Investigation of Homogeneous Equations Constructed from the Scattering Data

THEOREM 3.5.1: *If $F(t)$ is determined by the scattering data of he non-singular boundary-value problem* (2.1.1), i.e.,

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k| t} + F_s(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k| t} + \frac{1}{2\pi} \int_{-\infty}^{\infty} [I - S(\lambda)] e^{i\lambda t}\, d\lambda,$$

then

$$\text{(I)} \qquad x(t) + \int_0^\infty x(\xi) F(t+\xi)\, d\xi = 0, \qquad 0 \leq t < \infty$$

and

$$\text{(II)} \qquad \pm y(t) + \int_0^\infty y(\xi) F_s(-t-\xi)\, d\xi = 0, \qquad 0 \leq t < \infty$$

have no non-trivial solutions in $\hat{L}_{(n)}(0, \infty)$, and the number of linearly independent solutions of

$$\text{(III)} \qquad z(t) + \int_0^\infty z(\xi) F_s(t+\xi)\, d\xi = 0, \qquad 0 \leq t < \infty$$

is equal to the sum of the ranks of the matrices M_k [in other words, the number of vector eigenfunctions of the problem (2.1.1)].

Proof: By Theorem 2.3.1 and Lemma 3.3.2, all solutions of the equations under consideration in $\hat{L}_{(n)}(0, \infty)$ also belong to $L^2_{(n)}(0, \infty)$.

By Lemma 3.4.1, the solutions of (II) and (III) will be exactly those $y(t) \in L^2_{(n)}(0, \infty)$ and $z(t) \in L^2_{(n)}(0, \infty)$, respectively, for which[17]

$$\begin{aligned} \pm \tilde{y}(\lambda) &= \tilde{y}(-\lambda) S(-\lambda), \\ \tilde{z}(\lambda) &= \tilde{z}(-\lambda) S(\lambda), \end{aligned} \tag{3.5.1}$$

---

[17] When $F_s(t)$ is replaced by $F_s(-t)$, as in equation (II), it is clearly necessary to replace $S(\lambda)$ by $S(-\lambda)$.

with

$$\tilde{y}(\lambda)=\int_0^\infty y(t)e^{-i\lambda t}\,dt, \qquad \tilde{z}(\lambda)=\int_0^\infty z(t)e^{-i\lambda t}\,dt$$

regular in the lower half-plane.

By now making use of the fact that the scattering matrix $S(\lambda)$ of the boundary-value problem (2.1.1) is given by (Cf. (2.3.3))

$$S(\lambda) = E^*(-\lambda)\,[E^*(\lambda)]^{-1} = E^{-1}(-\lambda)E(\lambda), \tag{3.5.2}$$

we find from (3.5.1) that

$$\pm\tilde{y}(\lambda)E^*(-\bar{\lambda}) = \tilde{y}(-\lambda)E^*(\bar{\lambda}) \qquad (\operatorname{Im}\lambda = 0), \tag{3.5.3}$$

$$\tilde{z}(\lambda)E^{-1}(\lambda) = \tilde{z}(-\lambda)E^{-1}(-\lambda) \qquad (\operatorname{Im}\lambda = 0). \tag{3.5.4}$$

The equation (3.5.3) shows that the function $\pm\tilde{y}(\lambda)E^*(-\bar{\lambda})$, which is regular in the halfplane $\operatorname{Im}\lambda < 0$,[18] is equal for real values of λ to the function $\tilde{y}(-\lambda)E^*(\bar{\lambda})$, regular in the upper halfplane. This implies that $\tilde{y}(\lambda)E^*(-\bar{\lambda})$ is an entire function, and since $E^*(-\bar{\lambda}) \to I$ and $\tilde{y}(\lambda) \to 0$ as $|\lambda| \to \infty$ ($\operatorname{Im}\lambda < 0$), it follows that $\tilde{y}(\lambda)E^*(-\bar{\lambda}) \equiv 0$. But then by Lemma 2.2.2, $\tilde{y}(\lambda) \equiv 0$, and as a result, $y(t) \equiv 0$.

Thus, we have shown that the only solution of (II) in $\hat{L}_{(n)}(0, \infty)$ is the trivial solution.

Let us now consider (3.5.4). By similar reasoning, the function

$$\varphi(\lambda) = \begin{cases} \tilde{z}(\lambda)E^{-1}(\lambda) & \text{for } \operatorname{Im}\lambda \leq 0 \\ \tilde{z}(-\lambda)E^{-1}(-\lambda) & \text{for } \operatorname{Im}\lambda \geq 0 \end{cases} \tag{3.5.5}$$

is meromorphic in the entire λ-plane, having only a finite number of simple poles at the points $\pm\lambda_k = \mp i\mu_k$, $\mu_k > 0$ ($k = 1,2,\ldots,p$), and, possibly, a singularity at $\lambda = 0$ (see § 2, Chapter II). Since by Lemma 2.4.1, $\lambda E^{-1}(\lambda)$ is bounded in the neighborhood of zero, the function $\lambda\varphi(\lambda)$ is regular at zero and in accordance with (3.5.5) is given by

$$\lambda\varphi(\lambda) = \lambda\left[\sum_{k=1}^{p} \frac{\tilde{z}(\lambda_k)N_k}{\lambda-\lambda_k} - \sum_{k=1}^{p} \frac{\tilde{z}(\lambda_k)N_k}{\lambda+\lambda_k}\right] + \psi(\lambda).$$

Here, N_k is the residue of $E^{-1}(\lambda)$ at the point $\lambda_k = -i\mu_k$, and $\psi(\lambda)$ is an entire function which grows less rapidly than $|\lambda|$. Therefore,

---

[18] $E^*(-\bar{\lambda})$ is regular, along with $E(\lambda)$, in the halfplane $\operatorname{Im}\lambda < 0$.

$\psi(\lambda) = c$, where c is a constant vector, and hence

$$\varphi(\lambda) = \sum_{k=1}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{z}(\lambda_k) N_k + \frac{c}{\lambda}.$$

But according to (3.5.4) and (3.5.5), $\varphi(-\lambda) = \varphi(\lambda)$ for real λ, and thus it follows that $c = 0$. As a result,

$$\varphi(\lambda) = \sum_{k=1}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{z}(\lambda_k) N_k$$

and this together with (3.5.5) gives

$$\tilde{z}(\lambda) = \sum_{k=1}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{z}(\lambda_k) N_k E(\lambda). \tag{3.5.6}$$

Thus, we have shown that the Fourier transform of any solution of (III) satisfies (3.5.6). Conversely, if we prescribe arbitrary vectors $\tilde{z}(\lambda_k)$, then a straightforward computation shows that the function $\tilde{z}(\lambda)$ determined by (3.5.6) satisfies the second equation of (3.5.1) and, therefore, is the Fourier transform of a solution of (III).

Hence, it follows that the number of linearly independent solutions of (III) is the same as the number of linearly independent vector functions of the form

$$\sum_{k=1}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} c^{(k)} N_k, \tag{3.5.7}$$

where $c^{(k)} = (c_1^{(k)}, \ldots c_n^{(k)})$ $(k = 1,2, \ldots, p)$ are arbitrary constant vectors. Therefore, as can easily be seen,[19] this number is equal to the sum of the ranks of the matrices M_k (see § 1, sub-section 2).

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[19] Denote by **R** the linear manifold of vector functions of the form (3. 5.7), and by $\mathbf{R}_k$ the linear manifolds of vector functions of the form $2\lambda_k(\lambda^2 - \lambda_k^2)^{-1} cN_k$ $(k = 1,2, \ldots, p)$, where c is any constant vector.

For any $f(\lambda) \in \mathbf{R}$, we have by definiton

$$f(\lambda) = \sum_{k=1}^{p} f^{(k)}(\lambda),$$

where

$$f^{(k)}(\lambda) \in \mathbf{R}_k.$$

Since from the identity $\sum_{k=1}^{p} f^{(k)}(\lambda) \equiv 0$ it clearly follows that $f^{(k)}(\lambda) \equiv 0$ for $k = 1,2, \ldots, p$, **R** is the direct sum of the linear manifolds $\mathbf{R}_k$. It is also clear that the dimension of $\mathbf{R}_k$ is the same as the rank of N_k and, as a result, the dimension of **R** is equal to the sum of the ranks of $N_1, N_2, \ldots, N_k$.

It still remains to prove that equation (I) has only the trivial solution. By the corollary to Lemma 3.4.1, every solution $x(t)$ of (I) belonging to $L^2_{(n)}(0, \infty)$ is simultaneously a solution of (III) and satisfies the conditions

$$\tilde{x}(\lambda_k)M_k = 0 \qquad (k = 1, 2, \ldots, p). \tag{3.5.8}$$

Since $x(t)$ is a solution of (III), its Fourier transform satisfies the relation (3.5.6), i.e.,

$$\tilde{x}(\lambda) = \sum_{k=1}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{x}(\lambda_k) N_k E(\lambda).$$

Finally, by the results of sub-section 2, §1 of this chapter, it follows that[20]

$$\tilde{x}(\lambda_k)N_k = 0 \qquad (k = 1, 2, \ldots, p).$$

Therefore, $\tilde{x}(\lambda) \equiv 0$ and hence, $x(t) \equiv 0$ also.

REMARK 1: The existence of only the trivial solution of (II) in $\hat{L}_{(n)}(0, \infty)$ is obviously equivalent to the existence of only the trivial solution of

$$\pm y(t) + \int_{-\infty}^{0} y(\xi) F_s(t + \xi)\, d\xi = 0, \qquad -\infty < t \leq 0$$

in $\hat{L}_{(n)}(-\infty, 0)$.

REMARK 2: In the investigation of equation (II) and the derivation of expression (3.5.6), we only made use of conditions 1° and 2° of §4 for $F_s(t)$ and the factorization of $S(\lambda)$ given by (3.5.2). The importance of this last formula is not the actual meaning of the matrix $E(\lambda)$ entering into the factorization but rather its analytical properties. If the above conditions hold for $F_s(t)$ and $S(\lambda)$, then by what was proved above, equation (II) has no non-trivial solutions, and the Fourier transform of any solution of (III) is given by

$$\tilde{z}(\lambda) = \sum_{k=1}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{z}(\lambda_k) N_k E(\lambda),$$

where $\lambda_k = -i\mu_k$, $\mu_k > 0$, are the poles of $E^{-1}(\lambda)$ and N_k the corresponding residues. Hence, the number of linearly independent solutions of (III) is equal to the sum of the ranks of the matrices N_k.

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[20] Since M_k and P_k have the same null space, (3. 5. 8) implies that $\tilde{x}(\lambda_k)P_k = 0$ for $k = 1,2,\ldots, p$ and hence using (3. 1. 7), we find that $\tilde{x}(\lambda_k)N_k = 0$, $k = 1,2,\ldots,p$.

CHAPTER IV

Parseval's Equality

§ 1. Preliminaries

1. Let $S(\lambda)$ be a unitary matrix with the following property:

I_s. The matrix $I—S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$ with rows belonging to $\hat{L}_{(n)}(-\infty, 0)$ and $L^1_{(n)}(0, \infty)$.

According to Theorem 2.3.1 and the remark of Theorem 3.2.1, I_s is certain to hold if $S(\lambda)$ is the scattering matrix of the non-singular boundary-value problem.

Let M_k and $\lambda_k = -i\mu_k$, $\mu_k > 0$, $(k = 1,2, \ldots, p)$ be any prescribed Hermitian matrices and numbers, respectively, and define

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-\mu_k t} + F_s(t). \tag{4.1.1}$$

By Theorem 3.4.1, the integral equation formed on the basis of $F(t)$, namely,

$$F(x+y)+K(x, y)+\int_x^\infty K(x, t)F(t+y)\,dt = 0, \qquad 0 < x \leq y \tag{4.1.2}$$

has a unique solution $K(x, y)$ with rows belonging to $L^1_{(n)}(x, \infty)$ for each fixed $x > 0$. By definition, $K(x, y) = 0$ when $x > y$.

Making use of (4.1.2), we shall presently show that the matrix functions

$$U(x, \lambda) = E(x, -\lambda)-E(x, \lambda)S(-\lambda) \qquad (0 < \lambda < \infty), \tag{4.1.3}$$

$$U(x, \lambda_k) = E(x, \lambda_k)M_k \qquad (k = 1, 2, \ldots, p), \tag{4.1.4}$$

where

$$E(x, \lambda) = e^{-i\lambda x}I + \int_x^\infty K(x, t)e^{-i\lambda t}\, dt, \tag{4.1.5}$$

satisfy Parseval's equality in a form equivalent to the following expansion of the δ-function:

$$\delta(x-y)I = \sum_{k=1}^{p} U(x, \lambda_k)U^*(y, \lambda_k) + $$

$$+\frac{1}{2\pi}\int_0^\infty U(x, \lambda)U^*(y, \lambda)\, d\lambda. \tag{4.1.6}$$

Conversely, we shall also show that if (4.1.6) holds, in which the matrices $U(x, \lambda)$ and $U(x, \lambda_k)$ are given by (4.1.3)—(4.1.5), then the matrix function $K(x, y)$ appearing in (4.1.5) is a solution of the fundamental equation (4.1.2), with $F(t)$ defined by (4.1.1).

These results imply, in particular, that if

$$S(\lambda); \quad \lambda_k^2,\ M_k \qquad (k=1,\ 2, \ldots, p)$$

is the scattering data for some non-singular boundary-value problem, then the expansion (4.1.6) is valid. This shows that besides having a discrete spectrum $\lambda_1^2, \lambda_2^2, \ldots, \lambda_p^2$, the non-singular problem also has a continuous spectrum comprising the entire positive axis, and that the columns of $U(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 1,2, \ldots, p)$ yield a complete set of normalized eigenfunctions for the problem.

2. To derive Parseval's equality, it is necessary to obtain some supplementary results, which we now proceed to do.

Let us denote by $L_{(n,n)}(\varepsilon,\infty)$ the normed linear space of all n-th order matrix functions summable over the interval (ε, ∞) with the norm

$$\|\Phi\|^{(\varepsilon)} = \int_\varepsilon^\infty |\Phi(t)|\, dt.$$

It is clear that if $\Phi(t)$ belongs to $L_{(n,n)}(\varepsilon, \infty)$, then its rows beong to $L^1_{(n)}(\varepsilon, \infty)$.

The function $F(t)$ given by (4.1.1) obviously belongs to $L_{(n,n)}(0, \infty)$ and generates in this space a family of linear operators $\mathbf{F}_x$ $(x > 0)$ defined by

$$\mathbf{F}_x[\Phi] = \int_0^\infty \Phi(t)F(2x+t+y)\, dt.$$

Since

$$\int_0^\infty |\mathbf{F}_x[\Phi]|\,dy \leq \int_0^\infty dy \int_0^\infty |\Phi(t)|\,|F(2x+t+y)|\,dt =$$
$$= \int_0^\infty |\Phi(t)|\,dt \int_{2x+t}^\infty |F(s)|\,ds \leq ||\Phi||^{(0)} \int_{2x}^\infty |F(s)|\,ds,$$

we have the following estimate for the norm of $\mathbf{F}_x$:

$$||\mathbf{F}_x|| \leq \int_{2x}^\infty |F(t)|\,dt.$$

From Theorem 3.4.1, it follows that a bounded inverse operator $(\mathbf{I}+\mathbf{F}_x)^{-1}$ exists for each $x > 0$, where $\mathbf{I}$ is the identity operator.

LEMMA 4.1.1: *For any $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ depending only on ε such that for all $x \geqq \varepsilon$*

$$||(\mathbf{I}+\mathbf{F}_x)^{-1}|| \leq c(\varepsilon) \qquad (x \geq \varepsilon).$$

Proof: Since $||\mathbf{F}_x|| \to 0$ as $x \to \infty$, we can find an $\alpha > \varepsilon$ such that $||\mathbf{F}_x|| < 1/2$ for $x > \alpha$, and therefore

$$||(\mathbf{I}+\mathbf{F}_x)^{-1}|| \leq \frac{1}{1-||\mathbf{F}_x||} < 2 \qquad (x > \alpha). \qquad (4.1.7)$$

Now, for any $x > 0$ and $x_0 > 0$, we have

$$\mathbf{I}+\mathbf{F}_x = (\mathbf{I}+\mathbf{F}_{x_0})\{\mathbf{I}+(\mathbf{I}+\mathbf{F}_{x_0})^{-1}(\mathbf{F}_x-\mathbf{F}_{x_0})\}.$$

As a result,

$$(\mathbf{I}+\mathbf{F}_x)^{-1} = \{\mathbf{I}+(\mathbf{I}+\mathbf{F}_{x_0})^{-1}(\mathbf{F}_x-\mathbf{F}_{x_0})\}^{-1}\cdot(\mathbf{I}+\mathbf{F}_{x_0})^{-1}\};$$
$$(\mathbf{I}+\mathbf{F}_{x_0})^{-1} = \{\mathbf{I}+(\mathbf{I}+\mathbf{F}_{x_0})^{-1}(\mathbf{F}_x-\mathbf{F}_{x_0})\}\,(\mathbf{I}+\mathbf{F}_x)^{-1}$$

and thus

$$||(\mathbf{I}+\mathbf{F}_x)^{-1}|| \leq ||\{\mathbf{I}+(\mathbf{I}+\mathbf{F}_{x_0})^{-1}(\mathbf{F}_x-\mathbf{F}_{x_0})\}^{-1}||\cdot||(\mathbf{I}+\mathbf{F}_{x_0})^{-1}||,$$
$$||(\mathbf{I}+\mathbf{F}_{x_0})^{-1}|| \leq ||\mathbf{I}+(\mathbf{I}+\mathbf{F}_{x_0})^{-1}(\mathbf{F}_x-\mathbf{F}_{x_0})||\cdot||(\mathbf{I}+\mathbf{F}_x)^{-1}||. \qquad (4.1.8)$$

Since for any $\Phi(t)$ belonging to $L_{(n,n)}(0,\infty)$

$$||(\mathbf{F}_x-\mathbf{F}_{x_0})[\Phi]||^{(0)} = \int_0^\infty \Big|\int_0^\infty \Phi(t)\,[F(2x+t+y)-$$
$$-F(2x_0+t+y)]\,dt\,\Big|\,dy \leq \int_0^\infty |\Phi(t)|\,dt \int_t^\infty |F(2x+s)-$$
$$-F(2x_0+s)|\,ds \leq ||\Phi||^{(0)}\cdot\int_0^\infty |F(2x+s)-F(2x_0+s)|\,ds,$$

we find that

$$\| \mathbf{F}_x - \mathbf{F}_{x_0} \| \leq \int_0^\infty | F(2x+s) - F(2x+s) | \, ds \tag{4.1.9}$$

Hence, $\|\mathbf{F}_x - \mathbf{F}_{x_0}\| \to 0$ as $x \to x_0$.

Therefore, for any $\delta > 0$ $(\delta < 1)$, we have

$$\| (\mathbf{I}+\mathbf{F}_{x_0})^{-1}(\mathbf{F}_x - \mathbf{F}_{x_0}) \| < \delta$$

for all x sufficiently close to x_0 $(x_0 > 0)$, and we conclude from (4.1.8) that

$$\| (\mathbf{I}+\mathbf{F}_x)^{-1} \| \leq \frac{1}{1-\delta} \| (\mathbf{I}+\mathbf{F}_{x_0})^{-1} \|;$$

$$\| (\mathbf{I}+\mathbf{F}_{x_0})^{-1} \| \leq (1+\delta) | \ (\mathbf{I}+\mathbf{F}_x)^{-1} \|.$$

This means that $\|(\mathbf{I}+\mathbf{F}_x)^{-1}\|$ is a continuous function of x for all positive values of x, and hence

$$\max_{\varepsilon \leq x \leq \alpha} \| (\mathbf{I}+\mathbf{F}_x)^{-1} \| = c(\varepsilon, \alpha)$$

exists. This together with (4.1.7) implies the result of the lemma with

$$c(\varepsilon) = \max \{c(\varepsilon, \alpha), 2\}.$$

Remark. If the operator $(\mathbf{I}+\mathbf{F}_0)^{-1}$ exists, then

$$\| (\mathbf{I}+\mathbf{F}_x)^{-1} \| \leq c_0 \qquad (x \geq 0),$$

where c_0 is some constant. This statement may be proved in exactly the same way as Lemma 4.1.1.

Let us now consider the fundamental equation (4.1.2). Making the substitution $t = \xi + x$ and $y = \eta + x$ yields

$$F(2x+\eta) + K(x, x+\eta) + \int_0^\infty K(x, x+\xi) F(2x+\xi+\eta) \, d\xi = 0.$$

This can also be expressed as

$$(\mathbf{I}+\mathbf{F}_x)[K(x, x+\eta)] = -F(2x+\eta).$$

By Lemma 4.1.1, we conclude from this that

$$\int_0^\infty | K(x, x+\eta) | \, d\eta = \int_x^\infty | K(x, y) | \, dy \leq$$

$$\leq \| (\mathbf{I}+\mathbf{F}_x)^{-1} \| \cdot \int_0^\infty | F(2x+\eta) | \, d\eta \leq c(\varepsilon) \int_{2x}^\infty | F(t) | \, dt \tag{4.1.10}$$

for all $x \geq \varepsilon > 0$. Similarly, for $K^*(y, z)$, we find that

$$\int_{y}^{\infty} |K^*(y, z)|\, dz \leq c(\varepsilon) \int_{2y}^{\infty} |F(t)|\, dt, \tag{4.1.10'}$$

for all $y \geq \varepsilon > 0$.

LEMMA 4.1.2: *For any* $\varepsilon > 0$, *there exists a constant* $c_1(\varepsilon)$ *depending only on* ε *such that*

$$\sup_{x, y \geq \varepsilon} \int_{-h}^{h} \left| F(x+y+t) + \int_{x}^{\infty} K(x, \xi) F(y+t+\xi)\, d\xi \right| dt \leq$$
$$\leq c_1(\varepsilon)\delta(\varepsilon, h),$$
$$\sup_{x, y \geq \varepsilon} \int_{-h}^{h} \left| F(x+y+t) + \int_{y}^{\infty} F(x+t+\eta) K^*(y, \eta)\, d\eta \right| dt \leq$$
$$\leq c_1(\varepsilon)\delta(\varepsilon, h),$$

where

$$\delta(\varepsilon, h) = \max_{u \geq 2\varepsilon} \int_{-h}^{h} |F(u+t)|\, dt \to 0$$

as $h \to 0$.

Proof: Let $x, y \geq \varepsilon > 0$. Then

$$\int_{-h}^{h} \left| F(x+y+t) + \int_{x}^{\infty} K(x, \xi) F(y+t+\xi)\, d\xi \right| dt \leq$$
$$\leq \int_{-h}^{h} |F(x+y+t)|\, dt + \int_{x}^{\infty} |K(x, \xi)|\, d\xi \int_{-h}^{h} |F(y+t+\xi)|\, dt \leq$$
$$\leq \max_{u \geq 2\varepsilon} \int_{-h}^{h} |F(u+t)|\, dt \left\{1 + \int_{x}^{\infty} |K(x, \xi)|\, d\xi\right\}$$

and by (4.1.10)

$$\sup_{x, y \geq \xi} \int_{-h}^{h} \left| F(x+y+t) + \int_{x}^{\infty} K(x, \xi) F(y+t+\xi)\, d\xi \right| dt \leq$$
$$\leq \max_{u \geq 2\varepsilon} \int_{-h}^{h} |F(u+t)|\, dt \left\{1 + c(\varepsilon) \int_{2\varepsilon}^{\infty} |F(s)|\, ds\right\} = c_1(\varepsilon)\delta(\varepsilon, h).$$

The second inequality is derived in a similar way. The fact that $\delta(\varepsilon, h) \to 0$ as $h \to 0$ is a consequence of $F(t)$ belonging to $L_{(n,\, n)}(0, \infty)$

COROLLARY: *For* $x,\ y \geq \varepsilon > 0$,

$$\begin{aligned}\sup_{x,y\geq\varepsilon} \int_{-h}^{h} |K(x,\ y+t)|\,dt &\leq c_1(\varepsilon)\delta(\varepsilon,\ h),\\ \sup_{x,y\geq\varepsilon} \int_{-h}^{h} |K^*(y,\ x+t)|\,dt &\leq c_1(\varepsilon)\delta(\varepsilon,\ h).\end{aligned} \tag{4.1.11}$$

In fact, since $K(x, y+t) = 0$ for $x > y+t$ and since for $x \leq y+t$

$$K(x,\ y+t) = -F(x+y+t) - \int_{x}^{\infty} K(x,\ \xi)F(y+t+\xi)\,d\xi,$$

the first inequality of (4.1.11) follows immediately from the above lemma. The second inequality is proved analogously.

§ 2. Derivation of Parseval's Equality from the Fundamental Equation

Let us recall, first of all, that the matrix $F_s(t)$ with the Fourier transform $I-S(\lambda)$ (property I_s) is Hermitian if and only if

$$S(-\lambda) = S^*(\lambda) = S^{-1}(\lambda)$$

(see footnote 14 of Chapter III).

Making use of this property of $S(\lambda)$ and the expression (4.1.5) for $E(x, \lambda)$, we immediately obtain from (4.1.3)

$$\begin{aligned}U(x,\ \lambda)U^*(y,\ \lambda) &= (e^{i\lambda(x-y)}+e^{-i\lambda(x-y)})I -\\ &- e^{-i\lambda(x+y)}S(-\lambda) - e^{i\lambda(x+y)}S(\lambda) +\\ &+ \int_{x}^{\infty} K(x,\ \xi)\left\{(e^{i\lambda(\xi-y)}+e^{-i\lambda(\xi-y)})I - e^{-i\lambda(\xi+y)}S(-\lambda) -\right.\\ &\left. - e^{i\lambda(\xi+y)}S(\lambda)\right\} d\xi + \int_{y}^{\infty} \left\{(e^{i\lambda(x-\eta)}+e^{-i\lambda(x-\eta)})I -\right.\\ &\left. - e^{-i\lambda(x+\eta)}S(-\lambda) - e^{i\lambda(x+\eta)}S(\lambda)\right\}K^*(y,\ \eta)\,d\eta +\\ &+ \int_{y}^{\infty}\int_{x}^{\infty} K(x,\ \xi)\left\{(e^{i\lambda(\xi-\eta)}+e^{-i\lambda(\xi-\eta)})I - e^{-i\lambda(\xi+\eta)}S(-\lambda) -\right.\\ &\left. - e^{i\lambda(\xi+\eta)}S(\lambda)\right\}K^*(y,\ \eta)\,d\xi\,d\eta.\end{aligned}$$

But

$$(e^{i\lambda(x-y)}+e^{-i\lambda(x-y)})I-e^{-i\lambda(x+y)}S(-\lambda)-e^{i\lambda(x+y)}S(\lambda)=$$
$$=4\sin\lambda x\sin\lambda yI+e^{-i\lambda(x+y)}\widetilde{F}_s(-\lambda)+e^{i\lambda(x+y)}\widetilde{F}_s(\lambda),$$

where

$$\widetilde{F}_s(\lambda)=I-S(\lambda)=\int_{-\infty}^{\infty}F_s(t)e^{-i\lambda t}\,dt.$$

Therefore

$$U(x,\ \lambda)U^*(y,\ \lambda)-4\sin\lambda x\sin\lambda y\,I=$$
$$=\widetilde{A}(x,\ y,\ \lambda)+\widetilde{A}(x,\ y,\ -\lambda),\tag{4.2.1}$$

where

$$\widetilde{A}(x,\ y,\ \lambda)=e^{i\lambda(x+y)}\widetilde{F}_s(\lambda)+\int_x^{\infty}K(x,\ \xi)\,\{(e^{i\lambda(\xi-y)}-$$
$$-e^{i\lambda(\xi+y)})I+e^{i\lambda(\xi+y)}\widetilde{F}_s(\lambda)\}\,d\xi+$$
$$+\int_y^{\infty}\{(e^{i\lambda(x-\eta)}-e^{i\lambda(x+\eta)})I+e^{i\lambda(x+\eta)}\widetilde{F}_s(\lambda)\}K^*(y,\ \eta)\,d\eta+$$
$$+\int_y^{\infty}\int_x^{\infty}K(x,\ \xi)\,\{(e^{i\lambda(\xi-\eta)}-e^{i\lambda(\xi+\eta)})I+$$
$$+e^{i\lambda(\xi+\eta)}\widetilde{F}_s(\lambda)\}K^*(y,\ \eta)\,d\xi\,d\eta.$$

We next multiply both sides of (4.2.1) by

$$\frac{1}{2\pi}\left(\frac{\sin\lambda h}{\lambda h}\right)^2\tag{*}$$

and integrate with respect to λ over the interval $(0,\ \infty)$. Since

$$\frac{1}{2\pi}\int_0^{\infty}\left(\frac{\sin\lambda h}{\lambda h}\right)^2[\widetilde{A}(\lambda)+\widetilde{A}(-\lambda)]\,d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\lambda h}{\lambda h}\right)^2\widetilde{A}(\lambda)\,d\lambda$$

and since (*) is the Fourier transform of the function

$$\varphi_h(t)=\begin{cases}\dfrac{1}{2\pi}\left(1-\dfrac{|t|}{2h}\right), & \text{if}\quad |t|\leq 2h\\ \qquad 0\qquad, & \text{if}\quad |t|>2h,\end{cases}$$

we can use the evenness of $\varphi_h(t)$ and the convolution theorem to show that

$$\frac{1}{2\pi}\int_0^\infty \{U(x,\ \lambda)U^*(y,\ \lambda)-4\sin\lambda x\sin\lambda yI\}\left(\frac{\sin\lambda h}{\lambda h}\right)^2 d\lambda =$$

$$= \int_{-2h}^{2h}\varphi_h(t)\,A(x,\ y,\ t)\,dt.$$

Here

$$\begin{aligned} A(x,\ y,\ t) &= F_s(x+y+t)+K(x,\ y+t)- \\ &-K(x,\ t-y)+K^*(y,\ x+t)-K^*(y,\ t-x)+ \\ &+\int_x^\infty K(x,\ \xi)F_s(\xi+y+t)\,d\xi+\int_y^\infty F_s(x+\eta+t)K^*(y,\ \eta)\,d\eta+ \\ &+\int_y^\infty K(x,\ \eta+t)K^*(y,\ \eta)\,d\eta-\int_y^\infty K(x,\ t-\eta)K^*(y,\ \eta)d\eta+ \\ &+\int_y^\infty\int_x^\infty K(x,\ \xi)F_s(\xi+\eta+t)K^*(y,\ \eta)\,d\xi\,d\eta. \end{aligned}$$

Now the integration with respect to t extends over the interval $(-2h, 2h)$ and by definition, $K(x, y) = 0$ for $x > y$. Thus, for any $\varepsilon \geq 0$ and for all $x, y \geq \varepsilon$ and $2h \leq \varepsilon$, we have

$$\frac{1}{2\pi}\int_0^\infty \{U(x,\ \lambda)U^*(y,\ \lambda)-4\sin\lambda x\sin\lambda yI\}\left(\frac{\sin\lambda h}{\lambda h}\right)^2 d\lambda =$$

$$\begin{aligned} &= \int_{-2h}^{2h}\varphi_h(t)\Big\{F_s(x+y+t)+K(x,\ y+t)+K^*(y,\ x+t)+ \\ &+\int_x^\infty K(x,\ \xi)F_s(\xi+y+t)\,d\xi+\int_y^\infty F_s(x+\eta+t)K^*(y,\ \eta)\,d\eta+ \\ &+\int_y^\infty K(x,\ \eta+t)K^*(y,\ \eta)\,d\eta+ \\ &+\int_y^\infty\int_x^\infty K(x,\ \xi)F_s(\xi+\eta+t)K^*(y,\ \eta)\,d\xi\,d\eta\Big\}\,dt. \qquad (4.2.2) \end{aligned}$$

From (4.1.4) and (4.1.5), it then follows that

$$U(x, \lambda_k)U^*(y, \lambda_k) = M_k^2 e^{-\mu_k(x+y)} + \int_x^\infty K(x, \xi) M_k^2 e^{-\mu_k(\xi+y)}\, d\xi +$$

$$+ \int_y^\infty M_k^2 K^*(y, \eta) e^{-\mu_k(x+\eta)}\, d\eta +$$

$$+ \int_y^\infty \int_x^\infty K(x, \xi) M_k^2 K^*(y, \eta) e^{-\mu_k(\xi+\eta)}\, d\xi\, d\eta.$$

Multiplying both sides of this equation by $e^{-\mu_k t}\varphi_h(t)$, summing the result from $k = 1$ to $k = p$, and then integrating with respect to t, we obtain

$$\sum_{k=1}^{p} U(x, \lambda_k)U^*(y, \lambda_k)\left(\frac{\sinh \mu_k h}{\mu_k h}\right)^2$$

$$= \int_{-2h}^{2h} \varphi_h(t) \left\{ \sum_{k=1}^{p} [M_k^2 e^{-\mu_k(x+y+t)} + \right.$$

$$+ \int_x^\infty K(x, \xi) M_k^2 e^{-\mu_k(\xi+y+t)}\, d\xi + \int_y^\infty M_k^2 K^*(y, \eta) e^{-\mu_k(x+\eta+t)}\, d\eta +$$

$$\left. + \int_y^\infty \int_x^\infty K(x, \xi) M_k^2 K^*(y, \eta) e^{-\mu_k(\xi+\eta+t)}\, d\xi\, d\eta] \right\} dt.$$

If this equation is added to (4.2.2) and (4.1.1) is applied, there results

$$\sum_{k=1}^{p} U(x, \lambda_k)U^*(y, \lambda_k)\left(\frac{\sinh \mu_k h}{\mu_k h}\right)^2 + \frac{1}{2\pi}\int_0^\infty [U(x, \lambda)U^*(y, \lambda) -$$

$$- 4 \sin \lambda x \sin \lambda y I]\left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda = R(x, y, h), \tag{4.2.3}$$

where

$$R(x, y, h) = \int_{-2h}^{2h} \varphi_h(t) \left\{ F(x+y+t) + K(x, y+t) + \right.$$

$$+ K^*(y, x+t) + \int_x^\infty K(x, \xi) F(\xi+y+t)\, d\xi +$$

$$+ \int_y^\infty F(x+\eta+t) K^*(y, \eta)\, d\eta + \int_y^\infty K(x, \eta+t) K^*(y, \eta)\, d\eta +$$

$$\left. + \int_y^\infty \int_x^\infty K(x, \xi) F(\xi+\eta+t) K^*(y, \eta)\, d\xi\, d\eta \right\} dt.$$

An immediate consequence of this formula and the fundamental equation (4.1.2) is that $R(x, y, h) = 0$ for $x,\ y \geq \varepsilon,\ 2h \leq \varepsilon$, and $|x-y| > 2h$. On the other hand, if $|x-y| \leq 2h$, then the Lemma 4.1.2, the estimates (4.1.10′) and (4.1.11), and the inequality $\varphi\ (t) \leq$ $\leq {}^1/_2 h$ show that

$$|\,R(x,\ y,\ h)\,| \leq \frac{1}{h}\, c_2(\varepsilon)\,\delta(\varepsilon,\ 2h).$$

Therefore, one can easily see that

$$\int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} \int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} |\,R(x,\ y,\ h)\,|\, dx\, dy \leq c_3(\varepsilon)\,\delta(\varepsilon,\ 2h), \tag{4.2.4}$$

where $c_3(\varepsilon)$ is a constant depending only on ε.

Now let $\Phi(t) = ||\varphi_{jk}(t)||_1^n$ be any matrix function whose elements are each continuous and vanish outside the interval $(\varepsilon, {}^1/\varepsilon)$. We multiply (4.2.3) on the left by $\Phi(x)$ and on the right by $\Phi^*(y)$ and then integrate with respect to x and y. If we set

$$U(\Phi;\ \lambda) = \int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} \Phi(t) U(t,\ \lambda)\, dt,$$

$$S(\Phi;\ \lambda) = \int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} \Phi(t) \sin \lambda t\, dt,$$

the result of the integration is expressible as

$$\sum_{h=1}^{p} U(\Phi;\ \lambda_k) U^*(\Phi;\ \lambda_k) \left(\frac{\sinh \mu_k h}{\mu_k h}\right)^2 +$$

$$+ \frac{1}{2\pi} \int\limits_0^{\infty} U(\Phi;\ \lambda) U^*(\Phi;\ \lambda) \left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda =$$

$$= \frac{2}{\pi} \int\limits_0^{\infty} S(\Phi;\ \lambda) S^*(\Phi;\ \lambda) \left(\frac{\sin \lambda h}{\lambda h}\right)^2 d\lambda +$$

$$+ \int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} \int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} \Phi(x) R(x,\ y,\ h) \Phi^*(y)\, dx\, dy.$$

Finally, using (4.2.4) and Parseval's equality for the Fourier sine transform, we find from this that as $h \to 0$

$$\sum_{k=1}^{p} U(\Phi;\, \lambda_k)U^*(\Phi;\, \lambda_k) + \\ + \frac{1}{2\pi} \int_0^\infty U(\Phi;\, \lambda)U^*(\Phi;\, \lambda)\, d\lambda = \int_0^\infty \Phi(t)\Phi^*(t)\, dt. \qquad (4.2.5)$$

Parseval's equality has thus been proved for continuous matrix functions vanishing outside some interval $(\varepsilon, {}^1/\varepsilon)$, where ε is an arbitrary positive quantity. However, as usual, from this follows the validity of (4.2.5) for any $\Phi(t)$ with rows in $L^2_{(n)}(0, \infty)$ and in this case

$$U(\Phi;\, \lambda) = \underset{\varepsilon \to 0}{\text{l.i.m.}} \int_\varepsilon^{\frac{1}{\varepsilon}} \Phi(t)U(t,\, \lambda)\, dt. \qquad (4.2.6)$$

If $\Phi(t)$ and $\psi(t)$ are any matrix functions with rows in $L^2_{(n)}(0, \infty)$, then by (4.2.5)

$$\sum_{k=1}^{p} U(\Phi;\, \lambda_k)U^*(\Psi;\, \lambda_k) + \frac{1}{2\pi} \int_0^\infty U(\Phi;\, \lambda)U^*(\Psi;\, \lambda)\, d\lambda = \\ = \int_0^\infty \Phi(t)\Psi^*(t)\, dt, \qquad (4.2.5')$$

in which $U(\psi;\, \lambda)$ is defined analogously to (4.2.6).

§ 3. Derivation of the Fundamental Equation from Parseval's Equality

Suppose that Parseval's equality (4.2.5), (4.2.6) is given and that the matrix functions $U(x, \lambda)$ therein are expressed by the formulas (cf. (4.1.3)—(4.1.5))

$$U(x,\, \lambda) = U_0(x,\, \lambda) + \int_x^\infty K(x,\, t)U_0(t,\, \lambda)\, dt \qquad (\lambda > 0), \qquad (4.3.1)$$

$$U(x,\, \lambda_k) = M_k e^{-\mu_k x} + \int_x^\infty K(x,\, t)M_k e^{-\mu_k t}\, dt \qquad (k = 1,\, 2, \ldots, p).$$

Here,

$$U_0(x,\, \lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda), \qquad (4.3.2)$$

$S(\lambda)$ is a unitary matrix with the property I_s, M_k is Hermitian' and $\lambda_k = -i\,\mu_k$, $\mu_k > 0$ $(k = 1, 2,\ldots,p)$. We now show that the matrix $K(x, y)$ appearing in (4.3.1) is a solution of the fundamental equation (4.1.2) with $F(t)$ defined by (4.1.1).

Concerning $K(x, y)$, we merely make the following apriori assumptions:

(*1*) $K(x, y) = 0$ for $x > y$;

(*2*) $K(x, y)$ is bounded and measurable in the domain $0 < \varepsilon \leq x \leq y$ for any $\varepsilon > 0$;

(*3*) for every fixed $x > 0$, $K(x, y)$ belongs to $L_{(n,n)}(x, \infty)$, and for any $\varepsilon > 0$, the function

$$\alpha(x) = \int_x^\infty |K(x, y)|\, dy$$

is bounded in the interval $\varepsilon \leq x < \infty$.

Suppose that $\Phi(t)$ and $\Psi(t)$ are any matrix functions vanishing outside the interval $(\varepsilon, {}^1/\varepsilon)$, $\varepsilon > 0$, and are bounded and measurable in this interval. It is easy to see that under our assumptions, the matrix functions

$$\Phi_K(t) = \int_0^\infty \Phi(x)K(x, t)\, dx, \quad \Psi_K(t) = \int_0^\infty \Psi(x)K(x, t)\, dx, \quad (4.3.3)$$

vanish for $t < \varepsilon$ and are bounded and measurable in the interval (ε, ∞).

Applying the first formula of (4.3.1), we obtain

$$U(\Phi;\ \lambda) = \int_0^\infty \Phi(x)U(x, \lambda)\, dx = \int_0^\infty \Phi(x)U_0(x, \lambda)\, dx + \\ + \int_0^\infty \Phi(x)\, dx \int_0^\infty K(x, t)U_0(t, \lambda)\, dt = \int_0^\infty \Phi(x)U_0(x, \lambda)\, dx + \\ + \int_0^\infty \Phi_K(t)U_0(t, \lambda)\, dt$$

(the interchange of the order of integration here is justified), i.e.,

$$U(\Phi;\ \lambda) = U_0(\Phi;\ \lambda) + U_0(\Phi_K;\ \lambda),$$

where

$$U_0(\Phi;\ \lambda) = \int_0^\infty \Phi(x)U_0(x, \lambda)\, dx,$$

$$U_0(\Phi_K;\ \lambda) = \int_0^\infty \Phi_K(t)U_0(t, \lambda)\, dt.$$

Similarly,

$$U(\Psi;\ \lambda) = U_0(\Psi;\ \lambda) + U_0(\Psi_K;\ \lambda).$$

Therefore,

$$\frac{1}{2\pi}\int_0^\infty U(\Phi;\ \lambda)U^*(\Psi;\ \lambda)\,d\lambda = \frac{1}{2\pi}\int_0^\infty U_0(\Phi;\ \lambda)U_0^*(\Psi;\ \lambda)\,d\lambda +$$
$$+\frac{1}{2\pi}\int_0^\infty U_0(\Phi_K;\ \lambda)U_0^*(\Psi;\ \lambda)\,d\lambda + \frac{1}{2\pi}\int_0^\infty U_0(\Phi;\ \lambda)U_0^*(\Psi_K;\ \lambda)\,d\lambda +$$
$$+\frac{1}{2\pi}\int_0^\infty U_0(\Phi_K;\ \lambda)U_0^*(\Psi_K;\ \lambda)\,d\lambda. \tag{4.3.4}$$

Next, (4.3.2) leads to

$$U_0(x,\ \lambda)U_0^*(y,\ \lambda) =$$
$$= (e^{i\lambda(x-y)} + e^{-i\lambda(x-y)})I - e^{-i\lambda(x+y)}S(-\lambda) -$$
$$- e^{i\lambda(x+y)}S(\lambda) = 4\sin\lambda x \sin\lambda y I + e^{i\lambda(x+y)}[I - S(\lambda)] +$$
$$+ e^{-i\lambda(x+y)}[I - S(-\lambda)]$$

and therefore

$$\frac{1}{2\pi}\int_0^\infty U_0(\Phi;\ \lambda)U_0^*(\Psi;\ \lambda)\,d\lambda =$$
$$= \frac{1}{2\pi}\int_0^\infty d\lambda \int_0^\infty\int_0^\infty \Phi(x)\left\{4\sin\lambda x\sin\lambda y I + e^{i\lambda(x+y)}[I - S(\lambda)] +\right.$$
$$\left. + e^{-i\lambda(x+y)}[I - S(-\lambda)]\right\}\Psi^*(y)\,dx\,dy =$$
$$= \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty \Phi(x)\sin\lambda x\,dx\right)\left(\int_0^\infty \Psi(y)\sin\lambda y\,dy\right)^* d\lambda +$$
$$+\frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_0^\infty \Phi(x)e^{i\lambda x}\,dx\right)[I - S(\lambda)]\left(\int_0^\infty \Psi(y)e^{-i\lambda y}\,dy\right)^* d\lambda.$$

Making use of theorems on the convolution, the present Parseval equality and the one for the Fourier sine transform, we find from this that

$$\frac{1}{2\pi}\int_0^\infty U_0(\Phi;\ \lambda)U_0^*(\Psi;\ \lambda)\,d\lambda = \int_0^\infty \Phi(t)\Psi^*(t)\,dt + \\ + \int_0^\infty\int_0^\infty \Phi(x)F_s(x+y)\Psi^*(y)\,dx\,dy, \tag{4.3.5}$$

where $F_s(t)$ is the matrix having $I—S(\lambda)$ as its Fourier transform.

If (4.3.5) and the relations analogous to it are substituted in (4.3.4), the formulas stet (4.3.3) applied, and the order of integration interchanged (which is permissible under our assumptions), there results

$$\frac{1}{2\pi}\int_0^\infty U(\Phi;\ \lambda)U^*(\Psi;\ \lambda)\,d\lambda = \int_0^\infty \Phi(t)\Psi^*(t)\,dt + \\ + \int_0^\infty\int_0^\infty \Phi(x)\,\{\ldots\}\,\Psi^*(y)\,dx\,dy, \tag{4.3.6}$$

$$\{\ldots\} = F_s(x+y) + K(x,\ y) + \int_0^\infty K(x,\ t)F_s(t+y)\,dt + K^*(y,\ x) + \\ + \int_0^\infty F_s(x+t)K^*(y,\ t)\,dt + \int_0^\infty K(x,\ t)K^*(y,\ t)\,dt + \\ + \int_0^\infty\int_0^\infty K(x,\ \tau)F_s(\tau+t)K^*(y,\ t)\,d\tau\,dt.$$

The second formula of (4.3.1) then yields

$$\sum_{k=1}^{p} U(\Phi;\ \lambda_k)U^*(\Psi;\ \lambda_k) = \\ = \sum_{k=1}^{p}\left\{\int_0^\infty \Phi(x)\left[M_k e^{-\mu_k x} + \int_0^\infty K(x,\ t)M_k e^{-\mu_k t}\,dt\right]dx\right\}\times \\ \times\left\{\int_0^\infty\left[M_k\ e^{-\mu_k y} + \int_0^\infty M_k K^*(y,\ t)e^{-\mu_k t}\,dt\right]\Psi^*(y)\,dy\right\} = \\ = \int_0^\infty\int_0^\infty \Phi(x)\left\{\sum_{k=1}^{p} M_k^2 e^{-\mu_k(x+y)} + \int_0^\infty K(x,\ t)\sum_{k=1}^{p} M_k^2 e^{-\mu_k(t+y)}\,dt + \right. \\ + \int_0^\infty \sum_{k=1}^{p} M_k^2 e^{-\mu_k(x+t)}K^*(y,\ t)\,dt + \\ \left. + \int_0^\infty\int_0^\infty K(x,\ \tau)\sum_{k=1}^{p} M_k^2 e^{-\mu_k(\tau+t)}K^*(y,\ t)\,d\tau\,dt\right\}\Psi^*(y)\,dx\,dy.$$

Combining this with (4.3.6) and applying the Parseval equality (4.2.5′), we obtain

$$0 = \int_0^\infty \int_0^\infty \Phi(x) \Big\{ F(x+y) + K(x, y) + \int_0^\infty K(x, t) F(t+y)\, dt +$$
$$+ K^*(y, x) + \int_0^\infty F(x+t) K^*(y, t)\, dt + \int_0^\infty K(x, t) K^*(y, t)\, dt +$$
$$+ \int_0^\infty \int_0^\infty K(x, \tau) F(\tau+t) K^*(y, t)\, d\tau\, dt \Big\} \Psi^*(y)\, dx\, dy, \qquad (4.3.7)$$

with

$$F(t) = F_s(t) + \sum_{k=1}^{p} M_k^2 e^{-\mu_k t}$$

(Cf. (4.1.1)). Since $\Phi(t)$ and $\Psi(t)$ are arbitrary matrices, it follows that the expression within the braces in the integrand of (4.3.7) is equal to zero. Noting further that $K(x, y) = 0$ if $x > y$, we thus have for $0 < x < y$:

$$F(x+y) + K(x, y) + \int_x^\infty K(x, t) F(t+y)\, dt +$$
$$+ \int_y^\infty \left[F(x+t) + K(x, t) + \int_x^\infty K(x, \tau) F(\tau+t)\, d\tau \right] K^*(y, t)\, dt = 0.$$

This means that for fixed $x > 0$ and $y > x$, the expression

$$F(x+y) + K(x, y) + \int_x^\infty K(x, t) F(t+y)\, dt$$

is the solution of a homogeneous Volterra integral equation, and therefore

$$F(x+y) + K(x, y) + \int_x^\infty K(x, t) F(t+y)\, dt = 0, \qquad 0 < x < y,$$

q.e.d.

CHAPTER V

The Inverse Problem

§ 1. Statement of the Problem

In Chapters II and III, it was shown that the following five conditions hold for the scattering data

$$S(\lambda),\ \lambda_k^2,\ M_k \qquad (k=1,\ 2,\ldots,p)$$

of the non-singular boundary-value problem (2.1.1) (Theorems 2.3.1, 3.2.1, and 3.5.1):

I_s. The matrix $I-S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$ with rows belonging to $\hat{L}_{(n)}(-\infty,0)$ and $L^1\ (0,\infty)$ and

$$F_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[I-S(\lambda)]e^{i\lambda t}\,d\lambda. \tag{5.1.1}$$

II_s. The derivative $F_s'(t)$ exists for all positive values of t and

$$\int_0^{\infty} t\,|\,F_s'(t)\,|\,dt<\infty.$$

III_s. The equation

$$-x(t)+\int_{-\infty}^{0}x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad -\infty<t\leq 0$$

has no non-trivial solution in $\hat{L}_{(n)}(-\infty,\,0)$.

IV. The equation

$$x(t)+\int_0^{\infty}x(\xi)F(t+\xi)d\xi = 0, \qquad 0\leq t<\infty,$$

where

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k| t} + \frac{1}{2\pi} \int_{-\infty}^{\infty} [I - S(\lambda)] e^{i\lambda t} \, d\lambda,$$

has no non-trivial solution in $\hat{L}_{(n)}(0, \infty)$.

V. The number of linearly independent solutions of

$$x(t) + \int_0^{\infty} x(\xi) F_s(t+\xi) \, d\xi = 0, \qquad 0 \leq t < \infty$$

is equal to the sum of the ranks of the normalization matrices M_1, $M_2, \ldots, M_p$.

Note that the first three properties only concern the scattering matrix $S(\lambda)$; this is the reason for the subscript *s*.

In this chapter, the inverse problem will be solved. The definitive result to be obtained here is that the above five conditions are not only necessary but also sufficient for a unitary matrix $S(\lambda)$, Hermitian matrices M_k, and numbers λ_k^2 $(k = 1, 2, \ldots, p)$ to be the scattering data of some non-singular boundary-value problem (2.1.1).

In the course of the investigation, the role of each of the five enumerated characteristic properties of the scattering data will also be clarified.

Let there be given p arbitrary Hermitian matrices M_k, p negative numbers λ_k^2, and a unitary matrix $S(\lambda)$ having the property I_s.

In the preceding chapters, it was proved that if $F(t)$ is constructed from the scattering data according to (5.1.1) and (5.1.2), then the fundamental equation

$$F(x+y) + K(x, y) + \int_x^{\infty} K(x, t) F(t+y) \, dt = 0 \tag{5.1.3}$$

has a unique solution $K(x, y)$ belonging to $L_{(n,n)}(x, \infty)$ for each fixed $x > 0$, and

$$\begin{aligned} U(x, \lambda) &= E(x, -\lambda) - E(x, \lambda) S(-\lambda) \qquad (\lambda > 0), \\ U(x, \lambda_k) &= E(x, \lambda_k) M_k \qquad (k = 1, 2, \ldots, p), \end{aligned} \tag{5.1.4}$$

where

$$E(x, \lambda) = e^{-i\lambda x} I + \int_x^{\infty} K(x, t) e^{-i\lambda t} \, dt \tag{5.1.5}$$

are matrix functions for which Parseval's equality (4.2.5) holds.

We shall presently show that if $S(\lambda)$ has the properties I_s and II_s, then $U(x, \lambda)$ and $U(x, \lambda_k)$ are solutions of [1]

$$Y'' + \lambda^2 Y = V(x)Y, \qquad 0 < x < \infty,$$

in which

$$V(x) = -2 \frac{d}{dx} K(x, x), \tag{5.1.6}$$

and

$$\int_{\varepsilon}^{\infty} x \mid V(x) \mid dx < \infty$$

for any value of $\varepsilon > 0$. This inequality will also hold for $\varepsilon = 0$ under the added assumption that the given data has the property IV.

However, even in this case, $U(x, \lambda)$ and $U(x, \lambda_k)$ do not in general satisfy the boundary condition, i.e., they do not vanish at $x = 0$. The fulfillment of the boundary condition is assured by properties III_s and V.

§ 2. Estimates for the Matrix $K(x, y)$

Let $F(t)$ be a given matrix tending to zero when $t \to \infty$ and satisfying the following two conditions:

($A\varepsilon$) For all $t > 0$, $F'(t)$ exists and for any $\varepsilon > 0$

$$\int_{\varepsilon}^{\infty} t \mid F'(t) \mid dt < \infty;$$

($B\varepsilon$) For any $\varepsilon > 0$ and matrix $\Psi(x)$ in $L_{(n,n)}(\varepsilon, \infty)$, the equation

$$\Phi(x) + \int_{\varepsilon}^{\infty} \Phi(t) F(t+x)\, dt = \Psi(x)$$

has a unique solution in $L_{(n,n)}(\varepsilon, \infty)$.

These conditions are automatically satisfied if $F(t)$ is constructed from (5.1.1) and (5.1.2) in terms of given Hermitian matrices M_k, numbers $\lambda_k^2 < 0$, and a unitary matrix $S(\lambda)$ having the properties I_s and II_s (see Theorem 3.4.1). In this case, II_s immediately yields the stronger condition

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[1] If the elements of the potential matrix $V(x)$ are permitted to be generalized functions, then this result holds under the single assumption I_S.

$$(A_0) \qquad \int_0^\infty t\,|\,F'(t)\,|\,dt < \infty.$$

If, in addition, $S(\lambda)$, M_k and λ_k^2 have the property IV, then by the corollary to Lemma 3.3.1, $F(t)$ will also satisfy the following condition:

(B_0) For any $\varepsilon \geq 0$ and matrix $\Psi(x)$ belonging to $L_{(n,n)}(\varepsilon, \infty)$, the equation

$$\Phi(x) + \int_\varepsilon^\infty \Phi(t)F(t+x)dt = \Psi(x)$$

has a unique solution in $L_{(n,n)}(\varepsilon, \infty)$.

Since it follows from (A_ε) that

$$\int_\varepsilon^\infty |\,F(t)\,|\,dt \leq \int_\varepsilon^\infty dt \int_t^\infty |\,F'(s)\,|\,ds = \int_\varepsilon^\infty |\,F'(s)\,|\,ds \int_\varepsilon^s dt \leq$$

$$\leq \int_\varepsilon^\infty s\,|\,F'(s)\,|\,ds < \infty,$$

i.e., $F(t)$ belongs to $L_{(n,n)}(\varepsilon, \infty)$ for any $\varepsilon > 0$, the formula

$$\mathbf{F}_x[\Phi] = \int_0^\infty \Phi(t)F(2x+t+y)\,dt \qquad (5.2.1)$$

defines an operator in $L_{(n,n)}(0, \infty)$ for any $x > 0$.

Now, Lemma 4.1.1. evidently holds for the family of operators $\mathbf{F}_x[\Phi]$. By virtue of this, the solution $K(x, y)$ of the fundamental equation (5.1.3) for a given matrix $F(t)$ satisfies the inequality

$$\int_0^\infty |\,K(x,\ x+\eta)\,|\,d\eta \leq c(\varepsilon) \int_{2x}^\infty |\,F(t)\,|\,dt \qquad (x \geq \varepsilon > 0). \quad (4.1.10)$$

Let us set

$$K(x,\ x+\eta) = A(x,\ \eta); \qquad (5.2.2)$$

then this inequality becomes

$$\int_0^\infty |\,A(x,\ \eta)\,|\,d\eta \leq c(\varepsilon) \int_{2x}^\infty |\,F(t)\,|\,dt \qquad (x \geq \varepsilon > 0).$$

Now introducing the notation

$$\begin{aligned} \tau(x) &= \int_x^\infty |F'(t)|\,dt \qquad (x>0), \\ \tau_1(x) &= \int_x^\infty t\,|F'(t)|\,dt \qquad (x>0), \end{aligned} \tag{5.2.3}$$

and observing that

$$\int_{2x}^\infty |F(t)|\,dt \leq \int_{2x}^\infty s\,|F'(s)|\,ds = \tau_1(2x),$$

we obtain

$$\int_0^\infty |A(x,\eta)|\,d\eta = \int_x^\infty |K(x,y)|\,dy \leq c(\varepsilon)\tau_1(2x) \qquad (x \geq \varepsilon > 0). \tag{5.2.4}$$

Making use of this result and the inequality

$$|F(t)| \leq \int_t^\infty |F'(s)|\,ds = \tau(t),$$

we find from (5.1.3) that for $0 < \varepsilon \leq x \leq y$

$$\begin{aligned} |K(x,y)| \leq |F(x+y)| + \int_x^\infty |K(x,t)|\;|F(t+y)|\,dt &\leq \\ \leq \tau(x+y) + \tau(x+y)\int_x^\infty |K(x,t)|\,dt &\leq \\ \leq \tau(x+y)\,\{1 + c(\varepsilon)\tau_1(2x)\}&, \end{aligned}$$

i.e., for any $\varepsilon > 0$

$$|K(x,y)| \leq c_1(\varepsilon)\tau(x+y) \qquad (0 < \varepsilon \leq x \leq y), \tag{5.2.5}$$

$$\begin{gathered} |A(x,\eta)| = |K(x, x+\eta)| \leq c_1(\varepsilon)\tau(2x+\eta) \\ (0 < \varepsilon \leq x;\ \eta \geq 0), \end{gathered} \tag{5.2.6}$$

where $c_1(\varepsilon) = 1 + c(\varepsilon)\tau_1(2\varepsilon)$.

If $F(t)$ satisfies conditions (A_0) and (B_0), then we may also apply the remark of Lemma 4.1.1 and thus obtain the inequalities

$$|K(x,y)| \leq c_1\tau(x+y) \qquad (0 \leq x \leq y,\ y \neq 0), \tag{5.2.5'}$$

$$|A(x,\eta)| \leq c_1\tau(2x+\eta) \qquad (x \geq 0,\ \eta \geq 0,\ x+\eta \neq 0), \tag{5.2.6'}$$

where $c_1 = 1 + c_0\tau_1(0)$.

§ 3. Existence of the Derivatives of $K(x, y)$

As in the preceding section, we shall assume that $F(t)$ satisfies the conditions (A_ε) and (B_ε). Furthermore, for simplicity, we shall also assume that $F'(t)$ is continuous for positive values of t.

Lemma 5.3.1: *If $F(t)$ satisfies (A_ε) and (B_ε), then for each $x > 0$ and $y \geq x$, the partial derivative $K_y(x, y)$ exists and can be found from* (5.1.3) *by differentiation with respect to y, i.e.*

$$K_y(x, y) = -F'(x+y) - \int_x^\infty K(x, t)F'(t+y)\,dt. \qquad (5.3.1)$$

Proof: Assuming x to be a fixed positive quantity and making use of (5.2.5), we obtain for any $N > x$

$$\left| \int_N^\infty K(x, t)F'(t+y)\,dt \right| \leq c_1(x) \int_N^\infty \tau(x+t)\, |F'(t+y)|\, dt \leq$$

$$\leq c_1(x)\tau(N) \int_{N+y}^\infty |F'(s)|\, ds \leq c_1(x)\tau^2(N),$$

where $c_1(x)$ is a constant depending on x. From this estimate, it follows that the integral on the right-hand side of (5.3.1) is uniformly convergent for $y \geq x$, and this proves the validity of (5.3.1).

Remark 1. The expression (5.3.1) shows that $K_y(x, y)$ is continuous in y and that for any fixed value of $x > 0$, the matrix $K_y(x, y)$ belongs to $L_{(n,n)}(x, \infty)$.

Remark 2. If it is further assumed that $F(t)$ has a continuous second derivative for all $t > 0$ and that for any $\varepsilon > 0$

$$\int_\varepsilon^\infty |F''(t)|\, dt < \infty,$$

then it is possible to prove that $K_{yy}(x, y)$ exists and is continuous for $y \geq x > 0$, and that

$$K_{yy}(x, y) = -F''(x+y) - \int_x^\infty K(x, t)F''(t+y)\,dt. \qquad (5.3.2)$$

From this, it follows that the matrix $K_{yy}(x, y)$ belongs to $L_{(n,n)}(x, \infty)$ for each fixed value of $x > 0$.

Let us now consider the matrix $A(x, \eta)$ defined by (5.2.2). From (5.1.3), we find that

$$F(2x+\eta) + A(x, \eta) + \int_0^\infty A(x, \xi)F(2x+\xi+\eta)\,d\xi = 0. \qquad (5.3.3)$$

Lemma 5.3.2: *If* $F(t)$ *satisfies* (A_ε) *and* (B_ε), *then for any* $x > 0$ $\eta \geq 0$, $A_x(x, \eta)$ *exists and*

$$A_x(x, \eta) = -2F'(2x+\eta) - 2\int_0^\infty A(x, \xi)F'(2x+\xi+\eta)\,d\xi - \\ - \int_0^\infty A_x(x, \xi)F(2x+\xi+\eta)\,d\xi. \tag{5.3.4}$$

This expression is obtained by differentiating (5.3.3) *with respect to* x.

Proof: Starting with (5.3.3), we form the difference quotient

$$\frac{\Delta_x A(x, \eta)}{h} + \frac{\Delta_x F(2x+\eta)}{h} + \\ + \int_0^\infty \frac{\Delta_x A(x, \xi)}{h} F(2x+2h+\xi+\eta)\,d\xi + \\ + \int_0^\infty A(x, \xi)\frac{\Delta_x F(2x+\xi+\eta)}{h}\,d\xi = 0, \tag{5.3.5}$$

where

$$\Delta_x A(x, \eta) = A(x+h, \eta) - A(x, \eta),$$
$$\Delta_x F(2x+\eta) = F(2x+2h+\eta) - \\ - F(2x+\eta) = 2hF'(2x+2\theta h+\eta) \qquad (0 < \theta < 1).$$

Set

$$\Phi_h(x, \eta) = -\frac{\Delta_x F(2x+\eta)}{h} - \int_0^\infty A(x, \xi)\frac{\Delta_x F(2x+\xi+\eta)}{h}\,d\xi.$$

Now, for fixed $x > 0$ and $h \neq 0$ such that $x+h > 0$, the matrices $\Phi_h(x, \eta)$ and $h^{-1}\Delta_x A(x, \eta)$ belong to $L_{(n,n)}(0, \infty)$ (see (5.2.4)), and we can therefore write (5.3.5) in the form

$$(\mathbf{I}+\mathbf{F}_{x+h})\frac{\Delta_x A(x, \eta)}{h} = \Phi_h(x, \eta)$$

(see (5.2.1)), or

$$\frac{\Delta_x A(x, \eta)}{h} = (\mathbf{I}+\mathbf{F}_{x+h})^{-1}[\Phi_h(x, \eta)]. \tag{5.3.6}$$

Making use of the properties of $F(t)$ and the estimates (5.2.4) and (5.2.6), we can easily show that $\Phi_h(x, \eta)$ has the limit

$$\Phi_0(x, \eta) = -2F'(2x+\eta)-2\int_0^{\infty} A(x, \xi)F'(2x+\xi+\eta)\,d\xi$$

as $h \to 0$ both in the sense of ordinary convergence and convergence in the metric of $L_{(n,n)}(0, \infty)$ with respect to the variable η.

But for any $x > 0$, we have (see (4.1.9))

$$\| \mathbf{F}_{x+h}-\mathbf{F}_x \| \leq \int_0^{\infty} | F(2x+2h+t)-F(2x+t) |\, dt \to 0$$

as $h \to 0$, and using the easily verified equality

$$\begin{aligned}(\mathbf{I}+\mathbf{F}_{x+h})^{-1}-(\mathbf{I}+\mathbf{F}_x)^{-1} &= \\ = \{[\mathbf{I}+(\mathbf{I}+\mathbf{F}_x)^{-1}(\mathbf{F}_{x+h}-\mathbf{F}_x)]^{-1}-\mathbf{I}\}\,(\mathbf{I}+\mathbf{F}_x)^{-1},\end{aligned}$$

we can show that

$$\lim_{h\to 0} \| (\mathbf{I}+\mathbf{F}_{x+h})^{-1}-(\mathbf{I}+\mathbf{F}_x)^{-1} \| = 0. \tag{5.3.7}$$

These results together with (5.3.6) enable us to prove that as $h \to 0$, $h^{-1}\, \Delta_x A(x, \eta)$ converges to $(\mathbf{I}+\mathbf{F}_x)^{-1}[\Phi_0(x, \eta)]$ in the metric of $L_{(n,n)}(0, \infty)$. For,

$$\begin{aligned}&\| (\mathbf{I}+\mathbf{F}_{x+h})^{-1}[\Phi_h(x, \eta)]-(\mathbf{I}+\mathbf{F}_x)^{-1}[\Phi_0(x, \eta)] \| \leq \\ &\leq \| (\mathbf{I}+\mathbf{F}_{x+h})^{-1}-(\mathbf{I}+\mathbf{F}_x)^{-1} \| \cdot \| \Phi_h \|^{(0)}+ \\ &+ \| (\mathbf{I}+\mathbf{F}_x)^{-1} \| \cdot \| \Phi_h-\Phi_0 \|^{(0)},\end{aligned}$$

and the right-hand side of this inequality approaches zero as $h \to 0$.

Again considering our original equation (5.3.5), we see that the last three terms on the left-hand side have ordinary limits as h → 0. Therefore

$$\lim_{h\to 0} \frac{\Delta_x A(x, \eta)}{h} = A_x(x, \eta)$$

exists, and by the above results

$$A_x(x, \eta) = (\mathbf{I}+\mathbf{F}_x)^{-1}[\Phi_0(x, \eta)].$$

But this is equivalent to (5.3.4), and the lemma is proved.

REMARK 1. Since $A(x, 0) = K(x, x)$, we have, in effect, shown that $dK(x, x)/dx$ exists under the assumptions of Lemma 5.3.2. Setting

$\eta = 0$, we find from (5.3.4) that

$$\frac{d}{dx} K(x,\ x) = -2F'(2x) - 2\int\limits_0^\infty A(x,\ \xi)F'(2x+\xi)\,d\xi - \\ - \int\limits_0^\infty A(x,\ \xi)F(2x+\xi)\,d\xi. \tag{5.3.8}$$

REMARK 2. The proof of Lemma 5.3.2. shows that for any fixed $x > 0$, $\Delta_x A(x, \eta)$ approaches zero in the metric of $L_{(n,n)}(0, \infty)$ as $h \to 0$. Using this fact, the inequality $|F(t)| \leq \tau(t)$, and the estimate (5.2.6), we can easily show from (5.3.3) that $A(x, \eta)$ is a continuous function of x and η for $x > 0$ and $\eta \geq 0$. But

$$K(x,\ y) = A(x,\ y-x)$$

and is therefore a continuous function of x and y for $0 < x \leq y$.

By now considering (5.3.1), we can show that $K_y(x, y)$ is a continuous function of x and y for $0 < x \leq y$.

LEMMA 5.3.3: *Under the assumptions of the preceding lemma, for each* $x > 0$ *and* $y \geq x$, *the derivative* $K_x(x, y)$ *exists, is continuous in* x *and* y *and*

$$K_x(x,\ y) = -F'(x+y) + K(x,\ x)F(x+y) - \\ - \int\limits_x^\infty K_x(x,\ t)F(t+y)\,dt, \tag{5.3.9}$$

the last expression being obtained by differentiating (5.1.3) *with respect to* x.

Proof: By (5.2.2),[2] we have

$$\frac{\Delta_x A(x,\ \eta)}{h} = \frac{K(x+h,\ x+h+\eta) - K(x,\ x+\eta)}{h} = \\ = \frac{K(x+h,\ x+h+\eta) - K(x+h,\ x+\eta)}{h} + \\ + \frac{K(x+h,\ x+\eta) - K(x,\ x+\eta)}{h} = K_y(x+h,\ x+\eta+\theta h) + \\ + \frac{K(x+h,\ x+\eta) - K(x,\ x+\eta)}{h}.$$

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[2] See footnote 5 of Chapter II.

From the existence of $A_x(x, \eta)$ and the continuity of $K_y(x, y)$, it follows that

$$\lim_{h\to 0} \frac{K(x+h,\ x+\eta)-K(x,\ x+\eta)}{h} = K_x(x,\ y)\,|_{y=x+\eta}$$

exists ($x > 0$, $\eta \geq 0$). Moreover,

$$K_x(x,\ y)\,|_{y=x+\eta} = A_x(x,\ \eta)-K_y(x,\ y)\,|_{y=x+\eta}\,,$$

or

$$A_x(x,\ \eta)\,|_{\eta=y-x} = K_x(x,\ y)+K_y(x,\ y) \qquad (0 < x \leq y). \qquad (5.3.10)$$

Inserting the value $\eta = y—x$ in (5.3.4) and making useof (5.2.2) and (5.3.10), we find after some simple computations that

$$K_x(x,\ y)+K_y(x,\ y) = -2F'(x+y)+K(x,\ x)F(x+y)-$$
$$-\int_x^\infty K(x,\ t)F'(t+y)\,dt-\int_x^\infty K_x(x,\ t)F(t+y)\,dt.$$

This in conjunction with (5.3.1) yields the required equation (5.3.9). Now, make the substitution $t = x+\xi$, $y = x+\eta$ in (5.3.9) and introduce the notation

$$K_x(x,\ y)\,|_{y=x+\eta}= B(x,\ \eta),$$
$$-F'(2x+\eta)+K(x,\ x)F'(2x+\eta) = \Psi(x,\ \eta).$$

As a result, (5.3.9) becomes

$$B(x,\ \eta)+\int_0^\infty B(x,\ \xi)F(2x+\xi+\eta)\,d\xi = \Psi(x,\ \eta), \qquad (5.3.11)$$

or (see (5.2.1))

$$B(x,\ \eta)=(\mathbf{I}+\mathbf{F}_x)^{-1}[\Psi(x,\ \eta)]. \qquad (5.3.12)$$

The matrix $\Psi(x, \eta)$ can easily be shown to be continuous in x and η for $x > 0$, $\eta \geq 0$ and to depend continuously on the parameter x in the metric of $L_{(n,n)}(0, \infty)$. Since the operator $(\mathbf{I}+\mathbf{F}_x)^{-1}$ also depends continuously on x for positive x (see (5.3.7)), it follows from (5.3.12) that $B(x, \eta)$ is continuous in x in the metric of $L_{(n,n)}(0, \infty)$. As a result, (5.3.11) implies that $B(x, \eta)$ is a continuous function of x and η in the ordinary sense ($x > 0$, $\eta \geq 0$). Therefore, $K_x(x, y) = B(x, y—x)$ is continuous for $0 < x \leq y$ and belongs to $L_{(n,n)}(x, \infty)$.

REMARK. If it is further assumed that $F(t)$ has a continuous second derivative for all positive t and that for all $\varepsilon > 0$

$$\int_\varepsilon^\infty t\,|\,F''(t)\,|\,dt<\infty,$$

then $K_{xx}(x, y)$ exists and is continuous for all $x > 0$ and $y \geq x$. Furthermore, this derivative satisfies the equation

$$K_{xx}(x, y) = -F''(x+y) + \frac{d}{dx}K(x, x)F(x+y) + \\ + K(x, x)F'(x+y) + K_x(x, t)\,|_{t=x}F(x+y) - \\ - \int_x^\infty K_{xx}(x, t)F(t+y)\,dt, \tag{5.3.13}$$

which is obtained by formally differentiating (5.3.9) with respect to x. This can be proved in exactly the same way as is the existence of $K_x(x, y)$, since the equation (5.3.9) differs from (5.1.3) only in that it has the free term

$$-F'(x+y) + K(x, x)F(x+y)$$

instead of $-F(x+y)$, and under our assumptions, this term behaves analogously to $-F(x+y)$ in regards to differentiability and convergence in the metric of $L_{(n,n)}(x, \infty)$ with respect to y.

Finally, in the case in question, $K_{xx}(x, y)$ belongs to $L_{(n,n)}(x, \infty)$ for each fixed value of $x > 0$.

LEMMA 5.3.4: *Under the assumptions of Lemma* 5.3.2, *for any* $\varepsilon > 0$,

$$\left|\frac{d}{dx}K(x, x) + 2F'(2x)\right| \leq k(\varepsilon)\tau^2(2x) \qquad (x \geq \varepsilon), \tag{5.3.14}$$

in which $\tau(x)$ *is defined by* (5.2.3) *and* $k(\varepsilon)$ *is a constant depending on* ε. *Therefore, for any* $\varepsilon > 0$

$$\int_\varepsilon^\infty x\left|\frac{d}{dx}K(x, x)\right|dx < \infty. \tag{5.3.15}$$

Proof: From (5.3.4), it follows that

$$A_x(x, \eta) = (\mathbf{I} + \mathbf{F}_x)^{-1}[\Phi_0(x, \eta)],$$

where

$$\Phi_0(x, \eta) = -2F'(2x+\eta) - 2\int_0^\infty A(x, \xi)F'(2x+\xi+\eta)\,d\xi,$$

and by Lemma 4.1.1,

$$\int_0^\infty |A_x(x, \eta)|\,d\eta \leq c(\varepsilon)\int_0^\infty |\Phi_0(x, \eta)|\,d\eta \qquad (x \geq \varepsilon).$$

Since

$$\int_0^\infty |F'(2x+\eta)|\,d\eta = \int_{2x}^\infty |F'(t)|\,dt = \tau(2x),$$

and since by (5.2.4),

$$\int_0^\infty \left| \int_0^\infty A(x, \xi)F'(2x+\xi+\eta)\,d\xi \right| d\eta \leq$$

$$\leq \int_0^\infty |A(x, \xi)|\,d\xi \int_{2x}^\infty |F'(t)|\,dt \leq c(\varepsilon)\tau_1(2x)\tau(2x) \leq c(\varepsilon)\tau_1(2\varepsilon)\tau(2x)$$

for any $x \geq \varepsilon$, we obtain

$$\int_0^\infty |\Phi_0(x, \eta)|\,d\eta \leq 2[1+c(\varepsilon)\tau_1(2\varepsilon)]\tau(2x) = k_1(\varepsilon)\tau(2x).$$

As a consequence,

$$\int_0^\infty |A_x(x, \eta)|\,d\eta \leq c(\varepsilon)k_1(\varepsilon)\tau(2x) \qquad (x \geq \varepsilon).$$

Applying this estimate, as well as the inequalities $|F(t)| \leq \tau(t)$ and (5.2.6), we find from (5.3.8) that

$$\left| \frac{d}{dx} K(x, x) + 2F'(2x) \right| \leq$$

$$\leq 2c_1(\varepsilon)\tau(2x) \int_0^\infty |F'(2x+\xi)|\,d\xi + \tau(2x) \int_0^\infty |A_x(x, \xi)|\,d\xi \leq$$

$$\leq [2c_1(\varepsilon)+c(\varepsilon)k_1(\varepsilon)]\tau^2(2x) = k(\varepsilon)\tau^2(2x).$$

This completes the proof of (5.3.14). To prove (5.3.15), it clearly suffices to show that $x\,\tau^2(x)$ is summable over the interval (ε, ∞) for $\varepsilon > 0$. But

$$x\tau(x) = x \int_x^\infty |F'(t)|\,dt \leq \int_x^\infty t\,|F'(t)|\,dt = \tau_1(x)$$

and therefore

$$\int_\varepsilon^\infty x\tau^2(x)\,dx \leq \int_\varepsilon^\infty \tau_1(x)\tau(x)\,dx \leq \tau_1(\varepsilon) \int_\varepsilon^\infty \tau(x)\,dx \leq \tau_1^2(\varepsilon).$$

REMARK. If $F(t)$ satisfies conditions (A_0) and (B_0) of § 2, then by using the remark of Lemma 4.4.1 and the estimate (5.2.6′), we can deduce the stronger result

$$\left|\frac{d}{dx}K(x,\,x)+2F'(2x)\right| \leq k\tau^2(2x) \qquad (x>0),$$

where k is a constant. From this, it then follows that

$$\int_0^\infty x\left|\frac{d}{dx}K(x,\,x)\right|dx<\infty.$$

§ 4. Derivation of the Differential Equation

THEOREM 5.4.1: *Let $F(t)$ satisfy conditions (A_ε) and (B_ε) of* § 2 *and let $K(x, y)$ be the solution of the fundamental equation* (5.1.3). *Then*

$$E(x,\lambda) = e^{-i\lambda x}I+\int_x^\infty K(x,t)e^{-i\lambda t}\,dt \qquad (\operatorname{Im}\lambda \leq 0) \qquad (5.4.1)$$

is a solution of

$$Y''+\lambda^2 Y = V(x)Y, \qquad 0<x<\infty, \qquad (5.4.2)$$

where

$$V(x) = -2\frac{d}{dx}K(x,\,x), \qquad (5.4.3)$$

and for any $\varepsilon>0$

$$|V(x)-4F'(2x)| \leq C(\varepsilon)\tau^2(2x) \qquad (x\geq\varepsilon), \qquad (5.4.4)$$

$C(\varepsilon)$ being a constant depending on ε, and

$$\int_\varepsilon^\infty x\,|V(x)|\,dx<\infty. \qquad (5.4.5)$$

Proof: We first observe that (5.4.4) and (5.4.5) follow immediately from Lemma 5.3.4.

Let us now prove the main statement of the theorem. To this end, we first assume that $F(t)$ not only satisfies (A_ε) and (B_ε), but also has a continuous second derivative for all $t>0$ such that for any $\varepsilon>0$

$$\int_\varepsilon^\infty t\,|F''(t)|\,dt<\infty.$$

According to the remark 2 of Lemma 5.3.1 and the remark of Lemma 5.3.3, the partial derivatives $K_{xx}(x, y)$ and $K_{yy}(x, y)$ exist and are continuous for $0 < x \leq y$; furthermore, for each fixed $x > 0$, they belong to $L_{(n,n)}(x, \infty)$ and satisfy the respective equations (5.3.13) and (5.3.2).

Integrating (5.3.2) by parts, we obtain

$$K_{yy}(x, y)+F''(x+y)-K(x, x)F'(x+y)+$$
$$+K_t(x, t)\,|_{t=x}F(x+y)+\int_x^\infty K_{tt}(x, t)F(t+y)\,dt = 0.$$

Subtracting this from (5.3.13) and using (5.4.3), we then have

$$K_{xx}(x, y)-K_{yy}(x, y)+V(x)F(x+y)+$$
$$+\int_x^\infty [K_{xx}(x, t)-K_{tt}(x, t)]F(t+y)\,dt = 0.$$

Hence, replacing the term $V(x)F(x+y)$ using (5.1.3), we finally obtain

$$K_{xx}(x, y)-K_{yy}(x, y)-V(x)K(x, y)+$$
$$+\int_x^\infty [K_{xx}(x, t)-K_{tt}(x, t)-V(x)K(x, t)]F(t+y)\,dt = 0.$$

By the property (B_ε), we conclude that

$$K_{xx}(x, y)-K_{yy}(x, y)-V(x)K(x, y) = 0, \tag{5.4.6}$$

since for each fixed value of $x > 0$, the matrix

$$K_{xx}(x, y)-K_{yy}(x, y)-V(x)K(x, y)$$

is in $L_{(n,n)}(x, \infty)$.

Furthermore, under our assumptions,

$$\lim_{x+y\to\infty} K_x(x, y)=\lim_{x+y\to\infty} K_y(x, y)=0. \tag{5.4.7}$$

These results can easily be deduced from (5.3.1) and (5.3.9).

Thus, $K(x, y)$ satisfies the equation (5.4.6) and the conditions (5.4.7), and $V(x)$, moreover, satisfies (5.4.5). But then, by the remark of Theorem 1.3.1, the matrix $E(x, \lambda)$ determined by (5.4.1) is a solution of (5.4.2).

Now, suppose that $F(t)$ satisfies only (A_ε) and (B_ε). Then we can clearly find a sequence of functions $F_m(t)$ with the following properties:

(*1*) $F_m(t)$ is twice continuously differentiable for $t > 0$, and for any $\varepsilon > 0$

$$\int_{\varepsilon}^{\infty} t\,|F_m'(t)|\,dt < \infty, \quad \int_{\varepsilon}^{\infty} t\,|F_m''(t)|\,dt < \infty \qquad (m = 1, 2, \ldots);$$

(*2*) $F_m(t) \to F(t)$ as $m \to \infty$ uniformly in each interval $[a, b]$, $a > 0$;

(*3*) For each $\varepsilon > 0$

$$\lim_{m\to\infty} \int_{\varepsilon}^{\infty} t\,|F_m'(t) - F'(t)|\,dt = 0.$$

From property (*3*), it also follows that for any $\varepsilon > 0$

$$\lim_{m\to\infty} \|F_m - F\|^{(\varepsilon)} = \lim_{m\to\infty} \int_{\varepsilon}^{\infty} |F_m(t) - F(t)|\,dt = 0.$$

Consider the operators

$$\mathbf{F}_x[\Phi] = \int_0^{\infty} \Phi(t)F(2x+t+y)\,dt \qquad (x > 0),$$

$$\mathbf{F}_x^{(m)}[\Phi] = \int_0^{\infty} \Phi(t)F_m(2x+t+y)\,dt \qquad (x > 0),$$

defined in $L_{(n,n)}(0, \infty)$ (Cf.(5.2.1)). It can easily be shown that

$$\|\mathbf{F}_x^{(m)} - \mathbf{F}_x\| \leq \int_{2x}^{\infty} |F_m(s) - F(s)|\,ds$$

so that for any $\varepsilon > 0$

$$\lim_{m\to\infty} \|\mathbf{F}_x^{(m)} - \mathbf{F}_x\| = 0 \tag{5.4.8}$$

uniformly in x for $x \geq \varepsilon$.

Hence, for m sufficiently large, $(\mathbf{I}+\mathbf{F}_x^{(m)})^{-1}$ is a bounded operator in $L_{(n,n)}(0, \infty)$; we shall assume this to be true for all values of m.

By then applying (5.4.8) and Lemma 4.1.1 in connection with the obvious relation

$$(\mathbf{I}+\mathbf{F}_x^{(m)})^{-1} - (\mathbf{I}+\mathbf{F}_x)^{-1} =$$
$$= \{[\mathbf{I}+(\mathbf{I}+\mathbf{F}_x)^{-1}(\mathbf{F}_x^{(m)} - \mathbf{F}_x)]^{-1} - \mathbf{I}\}\,(\mathbf{I}+\mathbf{F}_x)^{-1},$$

we find that

$$\lim_{m\to\infty} \|(\mathbf{I}+\mathbf{F}_x^{(m)})^{-1}\| = \|(\mathbf{I}+\mathbf{F}_x)^{-1}\|$$

uniformly in x for $x \geq \varepsilon$. Therefore, if $m \geq m_0$, with m_0 sufficiently large, then by Lemma 4.1.1

$$\|(\mathbf{I}+\mathbf{F}_x^{(m)})^{-1}\| \leq c(\varepsilon)+1 \qquad (x \geq \varepsilon). \tag{5.4.9}$$

From the existence of $(\mathbf{I}+\mathbf{F}_x^{(m)})^{-1}$, it follows in particular that the equation obtained from (5.1.3) by replacing $F(t)$ by $F_m(t)$ has a unique solution in $L_{(n,n)}(x, \infty)$. Let us denote it by $K_m(x, y)$, and let us define

$$V_m(x) = -2\frac{d}{dx}K_m(x, x),$$

$$E_m(x, \lambda) = e^{-i\lambda x}I+\int_x^\infty K_m(x, t)e^{-i\lambda t}\,dt.$$

By the first part of the proof, $E_m(x, \lambda)$ is a solution of

$$E_m''(x, \lambda)+\lambda^2E_m(x, \lambda) = V_m(x)E_m(x, \lambda). \tag{5.4.10}$$

Let us next show that for any $\varepsilon > 0$,

$$\lim_{m\to\infty} \int_x^\infty |K_m(x, y)-K(x, y)|\,dy = 0 \tag{5.4.11}$$

uniformly in x for $x \geq \varepsilon$. From (5.1.3) and the analogous equation for $K_m(x, y)$, we have

$$\begin{aligned} K_m(x, y)-K(x, y)+\int_x^\infty [K_m(x, t)-K(x, t)]F_m(t+y)\,dt &= \\ = [F(x+y)-F_m(x+y)]+\int_x^\infty K(x, t)\,[F(t+y)-F_m(t+y)]dt. \end{aligned} \tag{5.4.12}$$

Making the substitution $t = x+\xi$, $y = x+\eta$, we obtain

$$\begin{aligned} A_m(x, \eta)-A(x, \eta)+\int_0^\infty [A_m(x, \xi)-A(x, \xi)]F_m(2x+\xi+\eta)\,d\xi &= \\ =\Phi_m(x, \eta), \end{aligned}$$

where

$$A(x, \eta) = K(x, x+\eta), \quad A_m(x, \eta) = K_m(x, x+\eta),$$

$$\begin{aligned} \Phi_m(x, \eta) = [F(2x+\eta)-F_m(2x+\eta)]+ \\ +\int_0^\infty A(x, \xi)\,[F(2x+\xi+\eta)-F_m(2x+\xi+\eta)]\,d\xi. \end{aligned}$$

or in operator notation,

$$(\mathbf{I}+\mathbf{F}_x^{(m)})[A_m(x,\eta)-A(x,\eta)] = \Phi_m(x,\eta).$$

This and (5.4.9) then yield

$$||A_m(x,\eta)-A(x,\eta)||_\eta^{(0)} \leq [c(\varepsilon)+1]\,||\Phi_m(x,\eta)||_\eta^{(0)} \qquad (x \geq \varepsilon)$$

for any $\varepsilon > 0$ and $m \geq m_0$. But it is easy to see that $\Phi_m(x,\eta)$ tends to zero in the metric of $L_{(n,n)}(0,\infty)$ as $m \to \infty$ uniformly in x for $x \geq \varepsilon$. Therefore,

$$\lim_{m\to\infty} ||A_m(x,\eta)-A(x,\eta)||_\eta^{(0)} = 0$$

uniformly in x for $x \geq \varepsilon$, and this is equivalent to (5.4.11).

Applying this result to (5.4.12), we then find that

$$\lim_{m\to\infty} K_m(x,y)=K(x,y)$$

uniformly in each finite domain of the form $0 < \varepsilon \leq x \leq y \leq a$.

In exactly the same way, a consideration of (5.3.4) and the analogous equation for $\frac{\partial}{\partial x} A_m(x,\eta)$, shows that for any $\eta \geq 0$

$$\lim_{m\to\infty} \frac{\partial}{\partial x} A_m(x,\eta)=\frac{\partial}{\partial x} A(x,\eta)$$

uniformly in each interval $0 < \varepsilon \leq x \leq a$. In particular, for $\eta = 0$

$$\lim_{m\to\infty} V_m(x)=V(x)$$

uniformly in each interval $0 < \varepsilon \leq x \leq a$.

By means of (5.4.11), it can also be shown that for $\varepsilon > 0$

$$\lim_{m\to\infty} E_m(x,\lambda)=E(x,\lambda) \qquad (\operatorname{Im}\lambda \leq 0)$$

uniformly in x for $x \geq \varepsilon$.

From (5.4.10), we now conclude that the sequence $E_m''(x,\lambda)$ is uniformly convergent in each interval $0 < \varepsilon \leq x \leq a$. Hence, $E''(x,\lambda)$ exists for all $x > 0$ and

$$E''(x,\lambda)+\lambda^2E(x,\lambda) = V(x)E(x,\lambda).$$

This is the required result.

Remark. If $F(t)$ satisfies conditions (A_0) and (B_0), then by the remark of Lemma 5.3.4, the matrix $V(x)$ defined by (5.4.3) satisfies

$$|V(x)-4F'(2x)| \leq c\tau^2(2x) \qquad (x > 0), \qquad (5.4.4')$$

where c is a constant and

$$\int_0^\infty x\,|V(x)|\,dx<\infty. \tag{5.4.5'}$$

Finally, if $F(t)$ is Hermitian, then so is $K(x, x)$ and hence $V(x)$. For, if $F(t)$ is Hermitian, (5.1.3) then yields

$$K(x, x) = -F(2x)-\int_x^\infty K(x, t)F(t+x)\,dt,$$

$$K^*(x, x) = -F(2x)-\int_x^\infty F(t+x)K^*(x, t)\,dt,$$

and it suffices to show that the integrals in these two expressions are equal.

Now, by (5.1.3)

$$\begin{aligned}F(t+x) &= -K(x, t)-\int_x^\infty K(x, s)F(s+t)\,ds =\\ &= -K^*(x, t)-\int_x^\infty F(s+t)K^*(x, s)\,ds,\end{aligned}$$

and therefore

$$\begin{aligned}\int_x^\infty K(x, t)F(t+x)\,dt &= -\int_x^\infty K(x, t)K^*(x, t)\,dt-\\ &\quad-\int_x^\infty\int_x^\infty K(x, t)F(s+t)K^*(x, s)\,ds\,dt,\end{aligned}$$

$$\begin{aligned}\int_x^\infty F(t+x)K^*(x, t)\,dt &= -\int_x^\infty K(x, t)K^*(x, t)\,dt-\\ &\quad-\int_x^\infty\int_x^\infty K(x, s)F(s+t)K^*(x, t)\,ds\,dt.\end{aligned}$$

This proves the result.

Theorem 5.4.1 is valid, in particular, if the prescribed data, determining $F(t)$ by means of (5.1.1) and (5.1.2), satisfies conditions I_s and II_s. If the further condition IV is fulfilled, then (5.4.4′) and (5.4.5′) will also hold for $V(x)$.

§ 5. Fulfillment of the Boundary Condition

Let the Hermitian matrices M_k, the numbers $\lambda_k = -i\mu_k$, $\mu_k > 0$ $(k = 1, 2, \ldots, p)$ and the unitary matrix $S(\lambda)$ be given quantities satisfying I_s and II_s.

Retaining the notation of § 1 of this chapter, we shall examine the matrices

$$U(x, \lambda) = E(x, -\lambda) - E(x, \lambda)S(-\lambda) =$$
$$= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + \int_x^\infty K(x, t)\,[e^{i\lambda t}I - e^{-i\lambda t}S(-\lambda)]\,dt$$
$$(\lambda > 0),$$
$$U(x, \lambda_k) = E(x, \lambda_k)M_k =$$
$$= \left[e^{-\mu_k x}I + \int_x^\infty K(x, t)e^{-\mu_k t}\,dt\right]M_k \qquad (k = 1, 2, \ldots, p)$$

(see (5.1.4) and (5.1.5)) determining Parseval's equality (4.2.5), and we shall characterize the conditions on the given data that assure the fulfillment of the boundary condition

$$U(0, \lambda) = 0 \qquad (\lambda > 0),$$
$$U(0, \lambda_k) = 0 \qquad (k = 1, 2, \ldots, p).$$

Suppose that in addition to I_s the given data also has the property IV. Then the fundamental equation (5.1.3) for $x = 0$, i.e.,

$$F(t) + K(0, t) + \int_0^\infty K(0, \xi)F(t+\xi)\,d\xi = 0$$

has a unique solution $K(0, t)$ in $L_{(n,n)}(0, \infty)$.

Substitution of the expression (5.1.2) for $F(t)$ yields

$$\sum_{k=1}^p M_k^2 e^{-\mu_k t} + F_s(t) + K(0, t) +$$
$$+ \sum_{k=1}^p \int_0^\infty K(0, \xi)e^{-\mu_k \xi}\,d\xi \cdot M_k^2 e^{-\mu_k t} + \int_0^\infty K(0, \xi)F_s(t+\xi)\,d\xi = 0,$$

which according to (5.1.5) can be written in the form

$$\sum_{k=1}^p E(0, \lambda_k)M_k^2 e^{-\mu_k t} + F_s(t) + K(0, t) +$$
$$+ \int_0^\infty K(0, \xi)F_s(t+\xi)\,d\xi = 0. \tag{5.5.1}$$

LEMMA 5.5.1: *If $x(t)$ is a solution of*

$$x(t)+\int_0^\infty x(\xi)F_s(t+\xi)d\xi = 0, \qquad 0 \leq t < \infty, \tag{5.5.2}$$

belonging to $L^1_{(n)}$ $(0, \infty)$, *then*

$$\int_0^\infty x(t)F_s(t)\,dt=0.$$

Proof: Since $F_s(t)$ belongs to $L_{(n,n)}(0, \infty)$, Lemma 3.3.2 implies that $x(t)$ also belongs to $L^\infty_{(n)}(0, \infty)$ and $L^2_{(n)}(0, \infty)$. By Lemma 3.4.1, we therefore have

$$x(t)+\int_0^\infty x(\xi)F_s(t+\xi)\,d\xi = x(-t), \qquad -\infty < t < \infty,$$

or

$$\int_0^\infty x(\xi)F_s(t+\xi)\,d\xi = x(-t)-x(t), \qquad -\infty < t < \infty.$$

Hence, we see that the integral

$$\int_0^\infty x(\xi)F_s(t+\xi)\,d\xi$$

is an odd function of t and, because of the properties of $x(t)$ and $F(t)$, is continuous for all real values of t.
But then

$$\int_0^\infty x(\xi)F_s(\xi)\,d\xi=0,$$

q.e.d.

Let us now take the complex conjugate transpose of both sides of (5.5.1). We then premultiply the result by any vector solution $x(t)$ of (5.5.2) and integrate over the interval $(0, \infty)$. This yields

$$\sum_{k=1}^{p} \tilde{x}(\lambda_k)M_k^2E^*(0, \lambda_k)+\int_0^\infty x(t)F_s(t)\,dt+$$

$$+\int_0^\infty \left[x(t)+\int_0^\infty x(\xi)F_s(t+\xi)\,d\xi\right]K^*(0, t)\,dt = 0,$$

where

$$\tilde{x}(\lambda)=\int_0^\infty x(t)e^{-i\lambda t}\,dt.$$

Hence, Lemma 5.5.1 shows that

$$\sum_{k=1}^{p} \tilde{x}(\lambda_k) M_k^2 E^*(0, \lambda_k) = 0 \tag{5.5.3}$$

for any solution $x(t)$ of (5.5.2).

THEOREM 5.5.1: *If the unitary matrix* $S(\lambda)$, *Hermitian matrices* M_k, *and numbers* $\lambda_k = -i\mu_k$, $\mu_k > 0$ $(k = 1, 2, \ldots, p)$ *are any given quantities with the properties* I_s, IV, *and* V, *then*

$$U(0, \lambda_k) = E(0, \lambda_k) M_k = 0 \qquad (k = 1, 2, \ldots, p) \tag{5.5.4}$$

and

$$F_s(t) + K(0, t) + \int_0^\infty K(0, \xi) F_s(t+\xi)\, d\xi = 0. \tag{5.5.5}$$

Proof: Let

$$x^{(1)}(t),\ x^{(2)}(t). \ldots x^{(r)}(t)$$

be a complete set of linearly independent solutions of (5.5.2), so that any solution of this equation can be expressed as

$$x(t) = \sum_{j=1}^{r} \gamma_j x^{(j)}(t).$$

Now consider the system of linear algebraic equations

$$\sum_{j=1}^{r} \gamma_j \tilde{x}^{(j)}(\lambda_k) M_k = 0 \qquad (k = 1, 2, \ldots p) \tag{5.5.6}$$

in the unknowns γ_j. By the corollary to Lemma 3.4.1, the existence of only the trivial solution of this system is equivalent to the existence of only the trivial solution of

$$x(t) + \int_0^\infty x(\xi) F(t+\xi)\, d\xi = 0, \qquad 0 \leq t < \infty,$$

i.e., is equivalent to property IV.

Since by assumption, property IV holds, the system (5.5.6) has no non-trivial solution, and therefore the rank of its coefficient matrix is equal to r (i.e., the number of unknowns).

Now consider the nonhomogeneous system

$$\sum_{j=1}^{r} \gamma_j \tilde{x}^{(j)}(\lambda_k) M_k = a^{(k)} M_k \qquad (k = 1, 2, \ldots, p), \tag{5.5.6'}$$

where $a^{(k)}$ are any given vectors. It is easy to see that the rank of the augmented matrix of this system cannot exceed the sum of the ranks of the matrices M_k which by property V is r.

But then the coefficient matrix and the augmented matrix of the system (5.5.6′) have the same rank, and this rank is equal to the number of unknowns. Thus, this system has a unique solution. Therefore, for any vectors $a^{(k)}$, one can find numbers γ_j such that a solution of (5.5.2),

$$x(t)=\sum_{j=1}^{r}\gamma_j x^{(j)}(t),$$

will satisfy the relations

$$\tilde{x}(\lambda_k)M_k=a^{(k)}M_k \qquad (k=1,\ 2,\ldots,p).$$

From this and (5.5.3), it follows that

$$\sum_{k=1}^{p}a^{(k)}M_k^2E^*(0,\ \lambda_k)=0$$

for any given vectors $a^{(k)}$. Therefore

$$aM_k^2E^*(0,\ \lambda_k)=0 \qquad (k=1,\ 2,\ldots,p),$$

for any vector a. But then, for any a and b

$$\left(aM_k^2E^*(0,\ \lambda_k),\ b\right)=0 \qquad (k=1,\ 2,\ldots,p)$$

and hence

$$\left(aM_k,\ bE(0,\ \lambda_k)M_k\right)=0 \qquad (k=1,\ 2,\ldots,p).$$

Setting $a=b\ E(0,\lambda_k)$ in the last equation, we conclude that $b\,E(0,\lambda_k)M_k=0$ $(k=1,2,\ldots,p)$, and since b is arbitrary, this implies that

$$E(0,\ \lambda_k)M_k=U(0,\ \lambda_k)=0 \qquad (k=1,\ 2,\ldots,p).$$

This result in conjuction with (5.5.1) immediately yields (5.5.5) and the theorem is proved.

THEOREM 5.5.2: *If the unitary matrix* $S(\lambda)$, *Hermitian matrices* M_k, *and numbers* $\lambda_k=-i\mu_k$, $\mu_k>0$ $(k=1,2,\ldots,p)$ *are any given quantities with the properties* I_S, III_S, IV, *and* V, *then, in addition to* (5.5.4), *we have*

$$U(0,\ \lambda)=I-S(-\lambda)+\int_0^{\infty}K(0,\ t)\left[e^{i\lambda t}I-e^{-i\lambda t}S(-\lambda)\right]dt=0 \qquad (\lambda>0).$$

Proof: Set

$$\Phi(t) = F_s(t)+K(0, t)-K(0, -t)+ \\ + \int_0^\infty K(0, \xi)F_s(t+\xi)\, d\xi. \tag{5.5.7}$$

Since $K(x, y) = 0$ for $y < x$, it follows from (5.5.5) and (5.5.7) that $\Phi(t) = 0$ for $t > 0$. From the form of $\Phi(t)$, it is also apparent that its rows belong to $\hat{L}_{(n)}(-\infty, \infty)$. Postmultiplying both sides of (5.5.7) by $F_s(t+x)$ and integrating with respect to t, we obtain

$$\int_{-\infty}^0 \Phi(t)F_s(t+x)\, dt = \int_{-\infty}^\infty F_s(t)F_s(t+x)\, dt+ \\ + \int_0^\infty K(0, t)F_s(t+x)\, dt - \int_{-\infty}^0 K(0, -t)F_s(t+x)\, dt+ \\ + \int_0^\infty K(0, \xi)\, d\xi \int_{-\infty}^\infty F_s(t+\xi)F_s(t+x)\, dt. \tag{5.5.8}$$

Since $I-S(\lambda)$ is the Fourier transform of $F_s(t)$, by theorems on the convolution,[3] the Fourier transform of

$$\int_{-\infty}^\infty F_s(t)F_s(t+x)\, dt$$

is

$$[I-S(-\lambda)][I-S(\lambda)] = I-S(-\lambda)-S(\lambda)+I$$

Hence,

$$\int_{-\infty}^\infty F_s(t)F_s(t+x)\, dt = F_s(-x)+F_s(x). \tag{5.5.9}$$

In a similar way, we find that

$$\int_{-\infty}^\infty F_s(t+\xi)F_s(t+x)\, dt = F_s(\xi-x)+F_s(x-\xi),$$

---

[3] See Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 1948, Theorems 41, 64, and 65.

and therefore

$$\int_0^\infty K(0,\ \xi)\,d\xi \int_{-\infty}^\infty F_s(t+\xi)F_s(t+x)\,dt = \int_0^\infty K(0,\ \xi)F_s(\xi-x)\,d\xi +$$

$$+\int_0^\infty K(0,\ \xi)F_s(x-\xi)\,d\xi = \int_0^\infty K(0,\ \xi)F_s(\xi-x)\,d\xi +$$

$$+\int_{-\infty}^0 K(0,\ -t)F_s(x+t)\,dt.$$

By (5.5.7),

$$\int_0^\infty K(0,\ \xi)\,d\xi \int_{-\infty}^\infty F_s(t+\xi)F_s(t+x)\,dt =$$

$$= \Phi(-x)+K(0,\ x)-K(0,\ -x)-F_s(-x)+ \tag{5.5.10}$$

$$+\int_{-\infty}^0 K(0,\ -t)F_s(x+t)\,dt.$$

Finally, by (5.5.7) again,

$$\int_0^\infty K(0,\ t)F_s(t+x)\,dt = \Phi(x)+K(0,\ -x)- \tag{5.5.11}$$

$$-K(0,\ x)-F_s(x).$$

Now, substitution of (5.5.9)—(5.5.11) in (5.5.8) gives

$$\int_{-\infty}^0 \Phi(t)F_s(t+x)\,dt = \Phi(-x)+\Phi(x)$$

and hence

$$-\Phi(x)+\int_{-\infty}^0 \Phi(t)F_s(t+x)\,dt = 0, \qquad -\infty < x \leq 0.$$

By assumption (property III_s), it then follows that $\Phi(x) \equiv 0$, i.e.,

$$F_s(t)+K(0,\ t)-K(0,\ -t)+$$

$$+\int_0^\infty K(0,\ \xi)F_s(t+\xi)\,d\xi = 0, \qquad -\infty < t < \infty.$$

Taking the Fourier transform of this equation and denoting

$$\widetilde{K}(\lambda) = \int_0^\infty K(0,\ t)e^{-i\lambda t}\,dt,$$

we obtain

$$I-S(\lambda)+\widetilde{K}(\lambda)-\widetilde{K}(-\lambda)+\widetilde{K}(-\lambda)[I-S(\lambda)] = 0,$$

or

$$I+\widetilde{K}(\lambda)-[I+\widetilde{K}(-\lambda)]S(\lambda) = 0.$$

If this is then multiplied on the right by $-S(-\lambda)$, there results

$$I+\widetilde{K}(-\lambda)-[I+\widetilde{K}(\lambda)]S(-\lambda) = 0,$$

i.e.,

$$I-S(-\lambda)+\int_0^\infty K(0, t)[e^{i\lambda t}I-e^{-i\lambda t}S(-\lambda)]\,dt = 0,$$

q.e.d.

Remark. In the proof of Theorem 5.5.2, besides I_s and III_s, use was made of (5.5.5) which is valid if, first, the fundamental equation (5.1.3) has a solution for $x = 0$, and, second, if (5.5.4) holds.

§ 6. Characteristic Properties of the Scattering Data and Scattering Matrix

1. The fundamental result stated in §1 of this chapter is an immediate consequence of Theorems 5.4.1, 5.5.1, 5.5.2 and the remark of Theorem 5.4.1, namely:

Conditions I_s, II_s, III_s, IV, and V are necessary and sufficient for any given unitary matrix $S(\lambda)$, Hermitian matrices M_k, and numbers $\lambda_k^2 < 0 (k = 1,2,\ldots,p)$ to be the respective scattering matrix, normalization matrices, and eigenvalues (i.e., scattering data) for some uniquely determined non-singular boundary-value problem (2.1.1).

To actually determine the potential from the scattering data, it is necessary to construct the matrix $F(t)$ using (5.1.1) and (5.1.2), then to compose the fundamental equation (5.1.3), and finally, after finding $K(x, y)$ from (5.1.3), to apply (5.1.6).

If there is no discrete spectrum, i.e., $M_k = 0$ for all k, then by Theorem 3.4.2, the fundamental equation can be solved by the method of successive approximations for any $x > 0$. We note one other case, in which the fundamental equation can be solved by elementary means. Namely, if the elements of $S(\lambda)$ are rational functions, then it is easy to see that the kernel $F(t+y)$ of the fundamental equation is degenerate.

Suppose now that only the unitary matrix $S(\lambda)$ is prescribed. Will $S(\lambda)$ be the scattering matrix for some non-singular boundary-value problem if it has the properties I_S, II_S, and III_S?

To answer this question, we prove the following preliminary lemma.

LEMMA 5.6.1: *For any r-dimensional subspace of* $L^2_{(n)}(0, \infty)$, *it is possible to choose r non-negative Hermitian matrices* M_k *and r positive numbers* μ_k *such that*

(1) if $x(t)$ *is any vector of the given subspace and*

$$\int_0^\infty x(t)e^{-\mu_k t}\,dt \cdot M_k = 0 \qquad (k = 1,\ 2, \ldots, r),$$

then $x(t) \equiv 0$;

(2) the sum of the ranks of the matrices $M_1, M_2, \ldots, M_r$ *is r.*

Proof: Choose some vector $x^{(1)}(t) \neq 0$ in the given subpace and a positive number μ_1 such that

$$\int_0^\infty x^{(1)}(t)e^{-\mu_1 t}\,dt \neq 0$$

(this is always possible). Let

$$x^{(1)}(t) = \{x_1^{(1)}(t),\ x_2^{(1)}(t), \ldots, x_r^{(1)}(t)\}$$

and suppose that

$$\int_0^\infty x_{i_1}^{(1)}(t)e^{-\mu_1 t}\,dt \neq 0.$$

Now, let $m^{(1)}$ denote the vector whose i_1-th component is 1 and remaining components are zero. We then have

$$\int_0^\infty \left(x^{(1)}(t),\ m^{(1)}e^{-\mu_1 t}\right) dt = \int_0^\infty x_{i_1}^{(1)}(t)e^{-\mu_1 t}\,dt \neq 0.$$

The vectors of the given subspace satisfying the condition

$$\int_0^\infty \left(x(t),\ m^{(1)}e^{-\mu_1 t}\right) dt = 0,$$

form an $(r-1)$-dimensional subspace. We now repeat the above construction for this subspace. We choose a non-zero vector $x^{(2)}(t)$ in it and a number $\mu_2 > \mu_1$, and we form the vector $m^{(2)}$ whose i_2-th component is 1 and remaining components are zero, so that

$$\int_0^\infty \left(x^{(2)}(t),\ m^{(2)}e^{-\mu_2 t}\right) dt \neq 0.$$

Continuing this process, we finally obtain the vectors $x^{(1)}(t)$, $x^{(2)}(t), \ldots, x^{(r)}(t)$ from the given subspace, the numbers $0 < \mu_1 < < \mu_2 < \ldots < \mu_r$, and vectors $m^{(1)}, m^{(2)}, \ldots, m^{(r)}$ each having a single i_k-th component of 1 and remaining components of zero. Moreover

$$\int_0^\infty \left(x^{(j)}(t),\ m^{(k)}e^{-\mu_k t}\right) dt \begin{cases} = 0, & \text{if} \quad j > k \\ \neq 0, & \text{if} \quad j = k. \end{cases} \tag{5.6.1}$$

The r vectors $x^{(j)}(t)$ are clearly linearly independent and therefore form a basis for the given subspace. Thus, if

$$x(t) = \sum_{j=1}^{r} \alpha_j x^{(j)}(t)$$

and

$$\int_0^\infty \left(x(t),\ m^{(k)}e^{-\mu_k t}\right) dt = 0 \qquad (k = 1,\ 2, \ldots, r),$$

then by (5.6.1), all $\alpha_j = 0$ and hence $x(t) \equiv 0$.

We next define $M_k (k = 1,2, \ldots, r)$ to be the matrix having all zero elements except for the element at the intersection of its i_k-th row and i_k-th column which is 1. It is easy to show that the matrices M_k have the properties (*1*) and (*2*) stated in the lemma. In fact, if

$$\int_0^\infty x(t)e^{-\mu_k t}\, dt \cdot M_k = 0 \qquad (k = 1,\ 2, \ldots, r),$$

then

$$\int_0^\infty \left(x(t),\ m^{(k)}e^{-\mu_k t}\right) dt = 0 \qquad (k = 1,\ 2, \ldots, r).$$

and therefore $x(t) \equiv 0$. Since the rank of each matrix is 1, the sum of their ranks is r.

COROLLARY: *If $S(\lambda)$ is a unitary matrix with the property I_s, then it is always possible to find Hermitian matrices M_k and positive numbers μ_k such that the conditions IV and V are also satisfied.*

In fact, consider

$$x(t)+\int_0^\infty x(\xi)F_s(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty,$$

where $F_s(t)$ is given in terms of $S(\lambda)$ by (5.1.1). By Lemma 3.3.2, any solution of this equation in $\hat{L}_{(n)}(0, \infty)$ belongs to $L^2_{(n)}(0, \infty)$ also, and by the corollary to Lemma 3.3.3, the set of all such solutions forms a finite dimensional subspace of $L^2_{(n)}(0, \infty)$.

If this subspace has dimension r, then applying Lemma 5.6.1, we can find r Hermitian matrices M_k and r positive numbers μ_k such that the conditions (*1*) and (*2*) of that lemma are fulfilled. But condition (*2*) for the case in question corresponds exactly to condition *V*. Moreover, by the corollary to Lemma 3.3.1, it follows from condition (*1*) that

$$x(t)+\int_0^\infty x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty,$$

with

$$F(t) = \sum_{=}^{r} M_k^2 e^{-\mu_k t}+F_s(t),$$

has only the trivial solution in $\hat{L}_{(n)}(0, \infty)$, and hence, *IV* is satisfied.

In particular, we have proved

THEOREM 5.6.1: *Conditions I_s, II_s, and III_s are necessary and sufficient for a unitary matrix $S(\lambda)$ to be the scattering matrix for some non-singular boundary-value problem* (2.1.1).

2. To conclude this section, we shall sharpen condition I_s for $S(\lambda)$. Making use of the relation

$$S(\lambda) = E^{-1}(-\lambda)E(\lambda), \tag{5.6.2}$$

we showed in Theorem 2.3.1 that if $\det E(0) \neq 0$, then $I-S(\lambda)$ is the Fourier transform of a function summable over the entire real axis.

Suppose now that $\det E(0) = 0$, and let P_0 be a Hermitian matrix which is a projection onto the null space of $E(0)$, so that $P_0E(0) = 0$.

LEMMA 5.6.2: *The matrix $\lambda^{-1}P_0\,E(\lambda)$ is the Fourier transform of a matrix $P_0K_1(t)$ summable over the interval* $(0, \infty)$ *and equal to zero in the interval* $(-\infty, 0)$.

Proof: We begin with (5.5.5) (see also (3.1.4)):

$$F_s(y)+K(0, y)+\int_0^\infty K(0, t)F_s(t+y)\,dt = 0,$$

and we integrate it over the interval (z, ∞), $z \geq 0$. Denoting

$$K_1(z)=\int_z^\infty K(0, y)\,dy,$$

we obtain

$$\int_z^\infty F_s(y)dy+K_1(z)+\int_0^\infty K(0, t)\,dt\int_{z+t}^\infty F_s(u)\,du = 0.$$

Integration by parts then yields

$$\int_z^\infty F_s(y)\,dy+K_1(z)+K_1(0)\int_z^\infty F_s(u)\,du-$$
$$-\int_0^\infty K_1(t)F_s(z+t)\,dt = 0,$$

or

$$E(0)\int_z^\infty F_s(y)\,dy+K_1(z)-\int_0^\infty K_1(t)F_s(z+t)\,dt = 0, \qquad (5.6.3)$$

since by (5.1.5),

$$I+K_1(0) = I+\int_0^\infty K(0, t)\,dt = E(0, 0) = E(0).$$

Premultiplying (5.6.3) by P_0, we find that

$$P_0K_1(z)-\int_0^\infty P_0K_1(t)F_s(z+t)\,dt = 0. \qquad (5.6.4)$$

Now the matrix $K_1(z)$ is bounded in the interval $0 \leq z < \infty$ because $K(0, y)$ belongs to $L_{(n,n)}(0, \infty)$. According to the remark of Lemma 3.3.2, it therefore follows from (5.6.4) that $P_0K_1(z)$ belongs to $L_{(n,n)}(0, \infty)$. Hence,

$$\tilde{K}_1(\lambda)=\int_0^\infty P_0\,K_1(t)e^{-i\lambda t}\,dt$$

is the Fourier transform of a matrix $P_0K_1(t)$ that vanishes for $t< 0$ and belongs to $L_{(n,n)}(0, \infty)$.

If $\lambda \neq 0$, integration by parts then yields

$$\widetilde{K}_1(\lambda) = P_0 K_1(t)(-i\lambda)^{-1}e^{-i\lambda t}\Big|_0^\infty + (-i\lambda)^{-1}\int_0^\infty P_0 K(0,\, t)e^{-i\lambda t}\,dt =$$

$$= (i\lambda)^{-1}P_0\left[K_1(0) - \int_0^\infty K(0,\, t)e^{-i\lambda t}\,dt\right] =$$

$$= (i\lambda)^{-1}P_0\left[\int_0^\infty K(0,\, t)\,dt - \int_0^\infty K(0,\, t)e^{-i\lambda t}\,dt\right]$$

and by (5.1.5)

$$\widetilde{K}_1(\lambda) = (i\lambda)^{-1}P_0[E(0) - I - E(0,\, \lambda) + I] = (-i\lambda)^{-1}P_0 E(\lambda).$$

Thus, the statement of the lemma is proved.

The Fourier transform of a summable fuction is continuous over the entire real axis. Therefore, the matrix $\lambda^{-1}P_0 E(\lambda)$ is continuous for all real values of λ, and $\lim\limits_{\lambda\to 0} \lambda^{-1} P_0 E(\lambda)$ exists (we denote the limit by E_0):

$$\lim_{\lambda\to 0} \lambda^{-1}P_0 E(\lambda) = E_0. \tag{5.6.5}$$

From this and the relation $P_0^2 = P_0$, it obviously follows that

$$P_0 E_0 = E_0 \tag{5.6.6}$$

and thus

$$\operatorname{rank} E_0 \leq \operatorname{rank} P_0. \tag{5.6.7}$$

Furthermore, by the definition of P_0,

$$\operatorname{rank} P_0 = n - \operatorname{rank} E(0).$$

Since $S(\lambda)$ is unitary, the modulus of each of its elements does not exceed 1. Therefore, we can find a sequence of positive numbers $\lambda_n \to 0$ such that $\lim\limits_{n\to\infty} S(\lambda_n)$ and $\lim\limits_{n\to\infty} S(-\lambda_n)$ exist. We denote these respective limits by $S(\to +0)$ and $S(\to -0)$. We now define Q^+ and Q^- to be Hermitian matrices (operators) projecting on the respective ranges of the operators $[I - S(\to +0)]$ and $[I - S(\to -0)]$, so that

$$\begin{aligned} Q^\pm[I - S(\to \pm 0)] &= I - S(\to \pm 0); \\ \operatorname{rank} Q^\pm &= \operatorname{rank}[I - S(\to \pm 0)]. \end{aligned} \tag{5.6.8}$$

But inasmuch as $S(-\lambda) = S^*(\lambda)$ for $\lambda > 0$, it follows that $S(\to -0) = S^*(\to +0)$. Therefore, from (5.6.8) and the fact that Q^+ and Q^-

are Hermitian, we conclude that

$$I-S(\to\mp 0) = [I-S(\to\mp 0)]Q^{\pm}. \qquad (5.6.9)$$

LEMMA 5.6.3: *For real values of* λ, $D(\lambda) = [I-P_0+(i\lambda)^{-1}P_0]\,E(\lambda)$ *is a non-singular matrix,* i.e.

$$\det D(\lambda) = \det\{[I-P_0+(i\lambda)^{-1}P_0]E(\lambda)\} \neq 0.$$

Proof: By Lemma 2.2.2, det $E(\lambda) \neq 0$ for all real values of $\lambda \neq 0$. Therefore

$$\det D(\lambda) = \det[I-P_0+(i\lambda)^{-1}P_0]\cdot\det E(\lambda) \neq 0$$

for all real values of $\lambda \neq 0$. Thus, to complete the proof, it is necessary to show that

$$D(0) = \lim_{\lambda\to 0}[I-P_0+(i\lambda)^{-1}P_0]E(\lambda) = (I-P_0)E(0)-iE_0$$

(see (5.6.5)) is also non-singular.

To this end, we start with (2.3.2):

$$G(x,\lambda) = -(2i)^{-1}[E(x,-\lambda)-E(x,\lambda)S(-\lambda)]E^*(-\lambda)\,(-\lambda)^{-1},$$

where $G(x,\lambda)$ is the solution of (2.1.3) satisfying the conditions $G(0,\lambda)=0$ and $G'(0,\lambda)=I$. Postmultiplying this equation by P_0 and letting λ tend to zero through the sequences $\pm\lambda_n$, we obtain

$$G(x,0)P_0 = -(2i)^{-1}E(x,0)\,[I-S(\to\mp 0)]E_0^*. \qquad (5.6.10)$$

Since $\lim_{x\to 0} x^{-1}G(x,0) = I$, $G(x,0)\,P_0$ and P_0 will have the same rank for x sufficiently small. But the rank of a product of matrices does not exceed the rank of any of its factors. Therefore, (5.6.10) shows that for small values of x, rank $G(x,0)\,P_0 = \operatorname{rank} P_0 \leq \operatorname{rank} E_0^* = \operatorname{rank} E_0$ and

$$\operatorname{rank} G(x,0)P_0 = \operatorname{rank} P_0 \leq \operatorname{rank}[I-S(\to\mp 0)]. \qquad (5.6.11)$$

The first of these inequalities together with (5.6.7) yields

$$\operatorname{rank} P_0 = \operatorname{rank} E_0.$$

From (5.6.2) we further have

$$E(\lambda)-E(-\lambda)S(\lambda) = 0, \qquad (5.6.12)$$

and letting λ tend to zero through the sequences $\pm\lambda_n$, we obtain

$$E(0)\,[I-S(\to\pm 0)] = 0. \qquad (5.6.13)$$

This implies that

$$\operatorname{rank} P_0 \geq \operatorname{rank}[I-S(\to\pm 0)],$$

and hence by (5.6.11) and (5.6.8),

$$\operatorname{rank} P_0 = \operatorname{rank}[I-S(\rightarrow \pm 0)] = \operatorname{rank} Q^{\pm}\,. \tag{5.6.14}$$

From the definition of Q^{+} and Q^{-} and (5.6.13) and (5.6.14), it follows that

$$E(0)\,(I-Q^{\pm}) = E(0); \quad \operatorname{rank}(I-Q^{\pm}) = \operatorname{rank} E(0).$$

This means that both Q^{+} and Q^{-}, when applied on the right, are projections on the null space of $E(0)$. Therefore

$$Q^{+} = Q^{-} = Q$$

and $E(0)\,(I-Q) = E(0)$. Moreover, by the definition of P_0,

$$E(0) = (I-P_0)E(0) = (I-P_0)E(0)\,(I-Q) \tag{5.6.15}$$

while (5.6.9) leads to

$$[I-S(\rightarrow \mp 0)]Q = I-S(\rightarrow \mp 0). \tag{5.6.16}$$

Now, premultiplying (5.6.12) by $\lambda^{-1}P_0$ and letting λ tend to zero through the sequence λ_n, we find that

$$E_0 + E_0 S(\rightarrow +0) = 0$$

or

$$E_0\left\{I-\frac{1}{2}[I-S(\rightarrow +0)]\right\} = 0.$$

Next, postmultiplying this by $I-Q$ and then making use of (5.6.16), we obtain

$$E_0(I-Q) = 0,$$

and this result in conjunction with (5.6.6) yields

$$E_0 = P_0 E_0 Q. \tag{5.6.17}$$

Thus, by (5.6.15) and (5.6.17), we have

$$D(0) = (I-P_0)E(0)-iE_0 = (I-P_0)E(0)\,(I-Q)-iP_0 E_0 Q.$$

But, P_0 and Q are projections, the rank of the first term on the right is equal to the rank $(I-P_0) = \operatorname{rank}(I-Q)$, and the rank of the second term is equal to the rank $P_0 = \operatorname{rank} Q$. Hence, it follows that the first term maps the subspace $(I-Q)R_n$ onto $(I-P_0)R_n$ and the second term maps QR_n onto P_0R_n, each in a one-to-one manner, and their sum maps the entire n-dimensional vector space

R_n into itself in a one-to-one manner. Therefore, $D(0)$ is a non-singular matrix, and this completes the proof of the lemma.

It is now possible to significantly refine condition I_s on the scattering matrix of the non-singular boundary-value problem.

Theorem 5.6.2: *If $S(\lambda)$ is the scattering matrix of some non-singular boundary-value problem, then $I-S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$ summable over the entire real axis.*

Proof: This property of $S(\lambda)$ was previously proved under the assumption that $\det E(0) \neq 0$ (Theorem 2.3.1). In this theorem, it was also shown that if $\det E(0) = 0$, then

$$I-S(\lambda) = I-S^{(1)}(\lambda)-S^{(2)}(\lambda),$$

where [see (2.3.10), (2.3.11), and (2.3.13)]

$$S^{(1)}(\lambda) = S(\lambda)\frac{1+p(\lambda)}{1+r(\lambda)}, \qquad S^{(2)}(\lambda) = S(\lambda)\left[1-\frac{1+p(\lambda)}{1+r(\lambda)}\right], \tag{5.6.18}$$

$$1+p(\lambda) = \det E(-\lambda), \qquad r(\lambda) = p(\lambda)[1-h(\lambda)], \tag{5.6.19}$$

$$h(\lambda) = \begin{cases} 1 & , \text{ if } |\lambda| < l \\ -|\lambda|+l+1, & \text{if } l \leq |\lambda| \leq l+1 \\ 0 & , \text{ if } |\lambda| > l+1, \end{cases}$$

and $I-S^{(1)}(\lambda)$ is the Fourier transform of a matrix summable over the entire real axis.

Therefore to prove the stated theorem, it is sufficient to show that $S^{(2)}(\lambda)$ is the Fourier transform of a matrix summable over the entire real axis.

From (5.6.2) and the definition of $D(\lambda)$ (see Lemma 5.6.3), it follows that

$$S(\lambda) = D^{-1}(-\lambda)[I-P_0+(-i\lambda)^{-1}P_0][I-P_0+(i\lambda)^{-1}P_0]^{-1}D(\lambda),$$

or

$$S(\lambda) = D^{-1}(-\lambda)[I-2P_0]D(\lambda).$$

Hence, by (5.6.18) and (5.6.19), we obtain

$$S^{(2)}(\lambda) = -D^{-1}(-\lambda)[I-2P_0]D(\lambda)\frac{p(\lambda)h(\lambda)}{1+r(\lambda)},$$

or

$$S^{(2)}(\lambda) = -D^{-1}(-\lambda)h(\lambda)[I-2P_0]D(\lambda)\frac{p(\lambda)}{1+r(\lambda)}.$$

But it is easy to see from Lemma 5.6.2 and the definitions of $p(\lambda)$, $r(\lambda)$, and $h(\lambda)$ that

$$D(\lambda)\frac{p(\lambda)}{1+r(\lambda)}$$

is the Fourier transform of a matrix summable over the entire real axis. It therefore remains to prove that $D^{-1}(-\lambda)h(\lambda)$ also possesses this property. To do this, we must show, as in the proof of Theorem 2.3.1, that det $D^{-1}(-\lambda)h(\lambda)$ is the Fourier transform of some function summable over the entire real axis. However, this assertion is a consequence of Lemmas 5.6.2, 5.6.3, and a local theorem of Wiener [12].[4]

With this, the proof of the theorem is complete.

COROLLARY: The scattering matrix of the non-singular boundary-value problem is uniformly continuous over the entire real axis. Furthermore, from (5.6.14),

$$\operatorname{rank} P_0 = \operatorname{rank}[I-S(0)]. \tag{5.6.20}$$

Therefore, det $E(0) \neq 0$ if and only if $S(0) = I$. Finally, it can easily be seen that the matrix $I-S(0)$, applied on the right, is a projection onto the null space of $E(0)$.

§ 7. Examples

The purpose of this section is twofold: to give some simple examples illustrative of the above method of constructing the potential from the scattering data and to prove the independence of the characteristic properties of the data. More precisely, several examples will be considered in which one of the properties III_s, IV, or V does not hold.

For simplicity, let us denote

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I-P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 1: Let

$$S(\lambda) = \frac{(\lambda+i)(\lambda+2i)}{(\lambda-i)(\lambda-2i)}; \quad \lambda_1^2 = -1, \ M_1^2 = 6$$

(we are considering the scalar case, $n = 1$).

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[4] See, also, Hardy, *Divergent Series*, Oxford, 1951, pg. 358, Theorem 229.

Then

$$1-S(\lambda) = -\frac{6i\lambda}{(\lambda-i)(\lambda-2i)} = -\frac{6}{i\lambda+1}+\frac{12}{i\lambda+2}$$

and hence

$$F_s(t) = \begin{cases} -6e^{-t}+12^{-2t}, & t>0 \\ \quad 0 \quad , & t<0, \end{cases}$$

$$F(t) = 12e^{-2t} \quad (t>0).$$

In this case, it is easy to verify that the five characteristic conditions all hold, and as a consequence, the given quantities constitute the scattering data for some non-singular scalar boundary-value problem.

Solving the fundamental equation

$$12e^{-2(x+y)}+K(x, y)+12e^{-2y}\int_x^\infty K(x, t)e^{-2t}\,dt = 0,$$

we find that

$$K(x, y) = -\frac{12e^{-2(x+y)}}{1+3e^{-4x}}.$$

Hence

$$V(x) = -2\frac{d}{dx}K(x, x) = -\frac{96e^{-4x}}{(1+3e^{-4x})^2}$$

and

$$E(x, \lambda) = e^{-i\lambda x}-\frac{12e^{-2x}}{1+3e^{-4x}}\int_x^\infty e^{-2t}e^{-i\lambda t}\,dt =$$

$$= e^{-i\lambda x}\left[1-\frac{12e^{-4x}}{(2+i\lambda)(1+3e^{-4x})}\right].$$

The normalized eigenfunctions are given by

$$U(x, \lambda) = E(x, -\lambda)-E(x, \lambda)S(-\lambda) =$$

$$= e^{i\lambda x}\left[1-\frac{12e^{-4x}}{(2-i\lambda)(1+3e^{-4x})}\right]-$$

$$-e^{-i\lambda x}\left[1-\frac{12e^{-4x}}{(2+i\lambda)(1+3e^{-4x})}\right]\frac{(\lambda-i)(\lambda-2i)}{(\lambda+i)(\lambda+2i)} \qquad (\lambda>0),$$

$$U(x, \lambda_1) = U(x, -i) = E(x, -i)M_1 =$$

$$= \sqrt{6}e^{-x}\left[1-\frac{12e^{-4x}}{3(1+3e^{-4x})}\right] = \frac{\sqrt{6}e^{-x}(1-e^{-4x})}{1+3e^{-4x}}$$

from which it is easy to see that

$$U(0, \lambda) = 0 \quad (\lambda > 0); \qquad U(0, \lambda_1) = 0.$$

Example 2: Let there be given the second-order unitary matrix

$$S(\lambda) = (I-P) + \left(\frac{\lambda - i}{\lambda + i}\right)^2 P$$

(there is no point spectrum).

We have

$$I - S(\lambda) = \frac{4i\lambda}{(\lambda+i)^2} P$$

and therefore

$$F_s(t) = \begin{cases} 0 \quad , & t < 0 \\ 4(1+t)e^t P, & t > 0, \end{cases}$$
$$F(t) = 0 \quad (t > 0).$$

The four conditions I_s, II_s IV, and V are satisfied. However, a straightforward computation shows that

$$-x(t) + \int_{-\infty}^{0} x(\xi) F_s(t+\xi)\, d\xi = 0, \qquad -\infty < t \leq 0,$$

has the non-trivial solution

$$x(t) = \{0,\ (1+t)e^t\} \qquad (t < 0),$$

and III_s does not hold. Now since $F(t) = 0$ for $t > 0$,

$$K(x, y) \equiv 0 \qquad (0 \leq x \leq y)$$

and therefore

$$V(x) \equiv 0,$$
$$E(x, \lambda) \equiv e^{-i\lambda x} I.$$

However, the solution of the corresponding differential equation,

$$U(x, \lambda) = E(x, -\lambda) - E(x, \lambda)S(-\lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) \quad (\lambda > 0),$$

does not satisfy the coundary condition at $x = 0$, i.e.,

$$U(0, \lambda) \neq 0.$$

Example 3: Let

$$S(\lambda) = \frac{(\lambda+i)(\lambda+2i)}{(\lambda-i)(\lambda-2i)}(I-P) + \left(\frac{\lambda-i}{\lambda+i}\right)^2 P.$$
$$\lambda_1^2 = -1, \quad M_1^2 = 6(I-P).$$

Then (see Examples 1 and 2)

$$I-S(\lambda) = -\frac{6i\lambda}{(\lambda-i)(\lambda-2i)}(I-P)+\frac{4i\lambda}{(\lambda+i)^2}P,$$

$$F_s(t) = \begin{cases} (-6e^{-t}+12e^{-2t})(I-P), & t>0 \\ 4(1+t)e^tP \quad , & t<0, \end{cases}$$

$$F(t) = 12e^{-2t}(I-P) \qquad (t>0).$$

As in the preceding example, III_s is not satisfied here.

From the fundamental equation, we obtain

$$K(x, y) = \frac{-12e^{-2(x+y)}}{1+3e^{-4x}}(I-P),$$

so that

$$V(x) = -\frac{96e^{-4x}}{(1+3e^{-4x})^2}(I-P),$$

$$E(x, \lambda) = e^{-i\lambda x}\left[1-\frac{12e^{-4x}}{(2+i\lambda)(1+3e^{-4x})}\right](I-P)+e^{-i\lambda x}P.$$

By direct verification, we can easily satisfy ourselves that

$$U(x, \lambda) = E(x, -\lambda)-E(x, \lambda)S(-\lambda) \qquad (\lambda>0)$$

does not vanish at $x = 0$. At the same time, the boundary condition is satisfied on the discrete spectrum, for

$$U(x, \lambda_1) = U(x, -i) = E(x, -i)M_1 = \frac{\sqrt{6}e^{-x}(1-e^{-4x})}{1+3e^{-4x}}(I-P)$$

and hence, $U(0, \lambda_1) = 0$.

This is related to the fact that only the condition III_s does not hold and illustrates the results of § 5 of this chapter.

Example 4: Given the data

$$S(\lambda) = \left(\frac{\lambda+i}{\lambda-i}\right)^2(I-P)+P; \quad \lambda_1^2 = -1, \ M_1^2 = P.$$

In this case,

$$I-S(\lambda) = -\frac{4i\lambda}{(\lambda-i)^2}(I-P),$$

$$F_s(t) = \begin{cases} 4(1-t)e^{-t}(I-P), & t>0 \\ 0 \quad , & t<0, \end{cases}$$

$$F(t) = 4(1-t)e^{-t}(I-P)+e^{-t}P \qquad (t>0),$$

so that the conditions I_s, II_s, and III_s hold. Condition V also holds since it can easily be verified that to within a constant factor

$$x(t) = \{te^{-t},\ 0\} \qquad (t > 0)$$

is the unique solution of

$$x(t) + \int_0^\infty x(\xi) F_s(t+\xi)\, d\xi = 0, \quad 0 \leq t < \infty.$$

But, by a direct verification, we can satisfy ourselves that this solution also satisfies the equation

$$x(t) + \int_0^\infty x(\xi) F(t+\xi)\, d\xi = 0,$$

and this means that the condition IV does not hold here.

The solution of the corresponding fundamental equation is

$$K(x,\ y) = \frac{4(x+y-1)e^{-(x+y)} + 4(x-y+1)e^{-3x-y}}{1-4xe^{-2x}-e^{-4x}}(I-P) - \\ - \frac{2e^{-(x+y)}}{2+e^{-2x}} P,$$

and hence

$$V(x) = v_{11}(x)(I-P) - \frac{16e^{-2x}}{(2+e^{-2x})^2} P,$$

where

$$v_{11}(x) = -2 \frac{d}{dx} \frac{4(2x-1)e^{-2x} + 4e^{-4x}}{1-4xe^{-2x}-e^{-4x}}.$$

A simple computation shows that in the neighborhood of the origin

$$v_{11}(x) = 6x^{-2} + \varphi(x),$$

where $\varphi(x)$ is continuous at $x = 0$, and therefore

$$\int_\varepsilon^\infty x\,|V(x)|\,dx < \infty$$

for any $\varepsilon > 0$. However, this inequality fails to hold when $\varepsilon = 0$, and this fact is related to the absence of property IV.

Example 5: Consider the scalar case ($n = 1$) in which

$$S(\lambda) = 1; \qquad \lambda_1^2 = -1,\ M_1^2 = 1,$$

so that

$$F_s(t) \equiv 0, \quad F(t) = e^{-t} \qquad (t > 0).$$

It is easy to show that all of the properties are fulfilled except V. From the fundamental equation, we have

$$K(x, y) = -\frac{2e^{-(x+y)}}{2+e^{-2x}}$$

and therefore

$$V(x) = -2\frac{d}{dx}K(x, x) = -\frac{16e^{-2x}}{(2+e^{-2x})^2},$$

$$E(x, \lambda) = e^{-i\lambda x} - \frac{2e^{-x}}{2+e^{-2x}} \int_x^\infty e^{-t}e^{-i\lambda t}\,dt =$$

$$= e^{-i\lambda x}\left[1 - \frac{2e^{-2x}}{(1+i\lambda)(2+e^{-2x})}\right].$$

This then yields

$$U(x, \lambda) = E(x, -\lambda) - E(x, \lambda)S(-\lambda) =$$

$$= 2i\sin\lambda x - \frac{2e^{-2x}}{2+e^{-2x}} \cdot \frac{2i(\sin\lambda x + \lambda\cos\lambda x)}{1+\lambda^2} \qquad (\lambda > 0),$$

$$U(x, \lambda_1) = U(x, -i) = \frac{2e^{-x}}{2+e^{-2x}}.$$

These functions satisfy the corresponding differential equation, but they do not vanish at $x = 0$. In fact,

$$U(0, \lambda) = -\frac{4i\lambda}{3(1+\lambda^2)} \qquad (\lambda > 0); \qquad U(0, -i) = \frac{2}{3}.$$

PART TWO

The Boundary-Value Problem with Singularities

CHAPTER VI

Special Transformation Operators

§ 1. Method of Investigation

In this section we shall give a very brief outline of one way of carrying out the spectral analysis of a boundary-value problem with specified singularities and of investigating the corresponding inverse problem. In so doing, we shall confine ourselves to a single differential equation (the scalar case).

In treating the non-singular boundary-value problem, we made use of transformation operators which enabled certain bounded solutions of

$$y'' - v(x)y + \lambda^2 y = 0 \quad (0 < x < \infty), \quad \int_0^\infty x \,|\, v(x) \,|\, dx < \infty \qquad (6.1.1)$$

to be expressed in terms of the solutions of the simplest equation of its kind, namely,

$$y'' + \lambda^2 y = 0 \qquad (0 < x < \infty).$$

This suggests the possibility of a similar treatment of a singular boundary-value problem such as

$$y'' - [v(x) + \alpha(\alpha+1)x^{-2}]y + \lambda^2 y = 0 \qquad (0 < x < \infty), \qquad (6.1.2)$$

$$y(0) = 0, \quad \int_0^\infty x \,|\, v(x) \,|\, dx < \infty$$

in which use is made of the solutions of the simplest equation of its kind,

$$y'' - \alpha(\alpha+1)x^{-2}y + \lambda^2 y = 0 \qquad (6.1.3)$$

(which can be expressed in terms of Bessel functions).

Actually, it is possible to show that certain solutions of (6.1.2), bounded at infinity, can be expressed in terms of the solutions of (6.1.3) by means of a formula analogous to (1.3.2) of Theorem 1.3.1. In this connection, the corresponding kernel satisfies an equation similar to the fundamental equation (3.1.12).

However, this approach leads to considerable analytical difficulties connected primarily with investigating the integral equations obtained. These difficulties are further compounded when matrix equations are considered.

When α is an integer, as in the cases of most interest, another approach is possible which reduces the entire investigation of a boundary-value problem such as (6.1.2) to one which is not singular. Here, the solution of the inverse problem leads to an integral equation exactly like the fundamental equation (3.1.12) studied in detail in the first part of the book. In this connection, we recall Example 4, § 7, Chapter V where by solving the fundamental equation, we were able to construct the matrix potential $V(x)$ which turned out to have a singularity of the form $6x^{-2}(I-P)$ in the neighborhood of zero.

In general, such a reduction to a non-singular boundary-value problem can be accomplished by means of special transformation operators.

We shall now show how these operators are obtained in the scalar case.

Let $y(x)$ and $y_1(x)$ be solutions of the respective equations

$$y''-q(x)y+\mu_0 y = 0, \quad y_1''-q_1(x)y_1+\mu_0 y_1 = 0 \qquad (a < x < b)$$

and $\varphi(x, \mu)$ any solution of

$$\varphi''-q(x)\varphi+\mu\varphi = 0 \quad (a < x < b). \tag{6.1.4}$$

Here, $q(x)$ and $q_1(x)$ are continuous functions for $a < x < b$, and μ and μ_0 are distinct numbers.

Let $W\{u(x), v(x)\}$ denote the Wronskian of $u(x)$ and $v(x)$:

$$W\{u(x), v(x)\} = u(x)v'(x)-u'(x)v(x)$$

and define

$$r(x, \mu) = y_1(x)W\{y(x), \varphi(x, \mu)\}(\mu_0-\mu)^{-1}.$$

From this and the identity

$$\frac{d}{dx}W\{y(x), \varphi(x, \mu)\} = (\mu_0-\mu)y(x)\varphi(x, \mu)$$

it follows that

$$r'' = y_1''(x)W\{y(x),\ \varphi(x,\ \mu)\}(\mu_0-\mu)^{-1}+2y_1'(x)y(x)\varphi(x,\ \mu)+$$
$$+y_1(x)[y(x)\varphi(x,\ \mu)]' = y_1''(x)W\{y(x),\ \varphi(x,\ \mu)\}(\mu_0-\mu)^{-1}+$$
$$+2[y_1(x)y(x)]'\varphi(x,\ \mu)+y_1(x)W\{y(x),\ \varphi(x,\ \mu)\}.$$

Hence, using the equation satisfied by $y_1(x)$, we obtain

$$r'' = [q_1(x)-\mu_0]y_1(x)W\{y(x),\ \varphi(x,\ \mu)\}(\mu_0-\mu)^{-1}+$$
$$+2[y_1(x)y(x)]'\varphi(x,\ \mu)+(\mu_0-\mu)y_1(x)W\{y(x),\ \varphi(x,\ \mu)\}(\mu_0-\mu)^{-1},$$

or

$$r''-q_1(x)r+\mu r = 2[y_1(x)y(x)]'\varphi(x,\ \mu). \tag{6.1.5}$$

Next, subtracting this equation from (6.1.4) and defining

$$\begin{aligned}\psi(x,\ \mu) &= \varphi(x,\ \mu)-r(x,\ \mu) = \\ &= \varphi(x,\ \mu)-y_1(x)W\{y(x),\ \varphi(x,\ \mu)\}(\mu_0-\mu)^{-1},\end{aligned} \tag{6.1.6}$$

we find that

$$\psi''-q_1(x)\psi+\mu\psi = \{q(x)-q_1(x)-2[y_1(x)y(x)]'\}\varphi(x,\ \mu). \tag{6.1.7}$$

Suppose first that $y(x)$ does not vanish for $a < x < b$, and choose

$$y_1(x) = y^{-1}(x).$$

Then, by (6.1.5),

$$r(x,\ \mu) = y^{-1}(x)W\{y(x),\ \varphi(x,\ \mu)\}(\mu_0-\mu)^{-1} \tag{6.1.8}$$

will satisfy the equation

$$r''-q_1(x)r+\mu r = 0, \tag{6.1.9}$$

where

$$q_1(x) = \mu_0+y_1''y_1^{-1} = \mu_0+[-y''y^{-2}+2(y')^2y^{-3}]y =$$
$$= \mu_0+y''y^{-1}-2[y''y^{-1}-(y'y^{-1})^2].$$

But $y''y^{-1} = q(x)-\mu_0$, so that

$$q_1(x) = q(x)-2[y'(x)y^{-1}(x)]'. \tag{6.1.10}$$

Thus, if $y(x)$ does not vanish in the interval $a < x < b$, then the operator defined by (6.1.8) transforms solutions $\varphi(x,\ \mu)$ of (6.1.4) into solutions $r(x,\ \mu)$ of (6.1.9), with $q_1(x)$ given by (6.1.10).

Now if the boundary-value problem defined by (6.1.1) and the condition $y(0) = 0$ has no negative spectrum, then the solution $\varphi(x)$ of (6.1.1) for $\lambda = 0$ which satisfies the initial conditions $\varphi(0) = 0$

and $\varphi'(0) = 1$ does not vanish in the interval $0 < x < \infty$. The normalized eigenfunctions $\varphi(x, \lambda)$ of this boundary-value problem and their derivatives will have the following asymptotic behavior at infinity:

$$\varphi(x, \lambda) \sim \sin\big(\lambda x + \delta(\lambda)\big), \quad \varphi'(x, \lambda) \sim \lambda \cos\big(\lambda x + \delta(\lambda)\big). \tag{6.1.11}$$

We now apply the transformation (6.1.8) to $\varphi(x, \lambda)$ taking $y(x) = \varphi(x)$ and $\mu = \lambda^2$. The function

$$\begin{aligned}\varphi_1(x, \lambda) &= -\lambda r(x, \lambda^2) = \lambda^{-1}\varphi^{-1}(x) W\{\varphi(x), \varphi(x, \lambda)\} = \\ &= \lambda^{-1}\varphi^{-1}(x)\,[\varphi(x)\varphi'(x, \lambda) - \varphi'(x)\varphi(x, \lambda)]\end{aligned} \tag{6.1.12}$$

then clearly satisfies the equation

$$\varphi_1'' - v_1(x)\varphi_1 + \lambda^2\varphi_1 = 0, \tag{6.1.13}$$

where

$$v_1(x) = v(x) - 2[\varphi'(x)\varphi^{-1}(x)]'. \tag{6.1.14}$$

Since as $x \to 0$

$$\varphi(x) = x[1+o(1)], \quad \varphi'(x) = 1+o(1), \quad \varphi(x, \lambda) = x[\varphi'(0, \lambda)+o(1)],$$

we have

$$\begin{aligned}\lim_{x\to 0} \varphi_1(x, \lambda) &= \lim_{x\to 0} \lambda^{-1}[\varphi'(x, \lambda) - \varphi^{-1}(x)\varphi'(x)\varphi(x, \lambda)] = \\ &= \lambda^{-1}[\varphi'(0, \lambda) - \varphi'(0, \lambda)] = 0,\end{aligned}$$

i.e.,

$$\varphi_1(0, \lambda) = 0 \qquad (0 < \lambda < \infty),$$

and

$$\begin{aligned}-2[\varphi'(x)\varphi^{-1}(x)]' &= -2[x^{-1} + o(x^{-1})]' = \\ &= 2x^{-2} + o(x^{-2}) \qquad (x \to 0).\end{aligned}$$

Hence, by (6.1.14), it follows that the potential $v_1(x)$ behaves like $2x^{-2}$ in the neighborhood of zero.

Now, as $x \to \infty$, $\varphi(x)$ and its derivative will have the asymptotic behavior

$$\varphi(x) = c + o(1), \quad \varphi'(x) = o(1), \tag{6.1.15}$$

if the boundary-value problem has a virtual level for $\lambda = 0$, or the behavior

$$\varphi(x) = x[c + o(1)], \quad \varphi'(x) = c + o(1), \tag{6.1.16}$$

if there exists no virtual level. From these relations, it follows that

$$-2[\varphi'(x)\varphi^{-1}(x)]' = -2[x^{-1}(1+o(1))]' =$$
$$= 2x^{-2}+o(x^{-2}) \qquad (x\to\infty),$$

when there is no virtual level, i.e., $v_1(x)$ has the same singularity $2x^{-2}$ at infinity.

However, if there is a virtual level for $\lambda = 0$, then it is not difficult to show that the new potential $v_1(x)$ will not be singular at infinity, i.e.,

$$\int_0^\infty x\,|\,v_1(x)\,|\,dx<\infty.$$

Finally, by (6.1.11), (6.1.12), (6.1.15), and (6.1.16), we find that

$$\varphi_1(x,\ \lambda) \sim \cos\left(\lambda x+\delta(\lambda)\right) = \sin\left(\lambda x+\frac{\pi}{2}+\delta(\lambda)\right) \qquad (x\to\infty).$$

Thus, the asymptotic phase $\delta(\lambda)$ of the eigenfunctions of the boundary-value problem described by the equation (6.1.1) and the boundary condition $y(0) = 0$ coincides with the asymptotic phase of the eigenfunctions of some problem with singularities of the form $2x^{-2}$ at zero and infinity (if no virtual level exists) or with such a singularity at zero only (if there is a virtual level).

The above procedure for introducing singularities may evidently be continued by starting with the equation (6.1.13) and repeating the argument.

In particular, when α is an integer n, it is simple to show in this way that if the boundary-value problem (6.1.2) has no negative spectrum and no virtual level for $\lambda = 0$, then a function $\delta_n(\lambda)$ may determine the asymptotic form of its eigenvalues, namely,

$$\varphi_n(x,\ \lambda) \sim \sin\left(\lambda x+n\frac{\pi}{2}+\delta_n(\lambda)\right) \qquad (x\to\infty)$$

if and only if it is the asymptotic phase of the eigenfunctions of a non-singular boundary-value problem.

We shall confine ourselves here to these general remarks on the usage of transformations of the form (6.1.8) in the investigation of singular problems.

It is clear that such transformations are not directly applicable when a negative spectrum or a virtual level exists.

Therefore, starting with formula (6.1.6), we shall introduce a transformation of a slightly different form which may be used in every case.

If we choose $y_1(x)$ in (6.1.6) so that

$$q(x)-q_1(x)-2[y_1(x)y(x)]' = 0, \tag{6.1.17}$$

that is,

$$y''(x)y^{-1}(x)-y_1''(x)y_1^{-1}(x)-2[y_1(x)y(x)]' = 0, \tag{6.1.18}$$

then by (6.1.7), $\psi(x, \mu)$ will satisfy

$$\psi''-q_1(x)\psi+\mu\psi = 0, \tag{6.1.19}$$

with

$$q_1(x) = q(x)-2[y_1(x)y(x)]'.$$

From (6.1.18), we have

$$[y'(x)y_1(x)-y(x)y_1'(x)]'y_1^{-1}(x)y^{-1}(x)-2[y_1(x)y(x)]' = 0,$$

or

$$[y'(x)y_1(x)-y(x)y_1'(x)]' = \{[y_1(x)y(x)]^2\}'.$$

Therefore, to satisfy condition (6.1.18), we must take $y_1(x)$ to be a solution of

$$y'(x)y_1(x)-y(x)y_1'(x) = [y_1(x)y(x)]^2+C_1,$$

where C_1 is a constant.

Now letting

$$y_1(x)=y(x)z(x),$$

we obtain the following equation for $z(x)$:

$$-y^2(x)z'(x) = z^2(x)y^4(x)+C_1.$$

In particular if $C_1 = 0$, then

$$-z^{-2}(x)z'(x) = y^2(x)$$

and therefore

$$z(x)=J^{-1}(x).$$

Here, $J(x)$ is any indefinite integral of $y^2(x)$.

Thus, if we choose

$$y_1(x)=y(x)J^{-1}(x), \qquad J'(x)=y^2(x)$$

in (6.1.6), then the function $\psi(x, \mu)$ obtained from $\varphi(x, \mu)$ by means of this transformation will satisfy (6.1.19) in which

$$q_1(x) = q(x) - 2[y^2(x)J^{-1}(x)]' = q(x) - 2[J'(x)J^{-1}(x)]'.$$

If $y(x)$ is square summable in the neighborhood of $x = a$ (or $x = b$) then a C_2 can always be chosen so that

$$J(x) = \int_a^x y^2(t)\,dt + C_2 \qquad \left(J(x) = \int_x^b y^2(t)\,dt + C_2\right)$$

does not vanish in the interval $a < x < b$.

We shall not stop here to clarify the role played by transformations of the form (6.1.6) in the investigation of the singular boundary-value problem since we shall elucidate this role in detail in the subsequent chapters where a particular example will be considered.

We wish to emhasize that (6.1.6) and (6.1.8) are not integral transformations,[1] and for precisely this reason they are applicable to any solution $\varphi(x, \mu)$ of equation (6.1.4). It is this property that permits an unrestricted investigation of the singular problem.

In the papers of M.M. Crum [13] and M.G. Krein [14], transformations of the form (6.1.8) were used conforming to solutions $\varphi(x, \mu)$ satisfying the same boundary condition as $y(x)$. In this case, the transformations can be expressed in integral form.

This relationship between the transformations used by Crum and Krein and those of the form (6.1.8) was brought to our attention by L.D. Faddeyev. He also observed that (6.1.6) can be formally represented as the composition of transformations of the form (6.1.8).

§ 2. Transformation Operators for Matrix Equations

In accordance with the above scheme, we shall now generalize the transformation operators (6.1.6) and (6.1.8) to the case of matrix equations.

Lemma 6.2.1: *Let $Z(x)$ and $Y_1(x)$ be matrices satisfying*

$$Z'' - ZQ(x) + \mu_0 Z = 0, \quad Y_1'' - Q_1(x)Y_1 + \mu_0 Y_1 = 0 \qquad (a < x < b)$$

---

[1] These transformations can be written in ordinary integral form for those solutions $\varphi(x,\mu)$ for which $W\{y(x), \varphi(x,\mu)\} = 0$ at $x = a$ or $x = b$ (see Lemma 6. 3. 1).

and $\Phi(x, \mu)$ *some solution of*

$$\Phi'' - Q(x)\Phi + \mu\Phi = 0 \qquad (a < x < b),$$

where $Q(x)$ *and* $Q_1(x)$ *are any square matrices continuous for* $a < x < b$, *and* μ *and* μ_0 *are distinct numbers.*

Then the matrices defined by

$$R(x, \mu) = Y_1(x)W\{Z(x),\ \Phi(x, \mu)\}(\mu_0 - \mu)^{-1}, \tag{6.2.1}$$

$$\Psi(x, \mu) = \Phi(x, \mu) - R(x, \mu) \tag{6.2.2}$$

satisfy

$$R'' - Q_1(x)R + \mu R = 2[Y_1(x)Z(x)]'\Phi(x, \mu), \tag{6.2.3}$$

$$\begin{aligned} &\Psi'' - Q_1(x)\Psi + \mu\Psi = \\ &= \{Q(x) - Q_1(x) - 2[Y_1(x)Z(x)]'\}\Phi(x, \mu) \end{aligned} \tag{6.2.4}$$

for $a < x < b$.

Proof: From the definition of $Z(x)$ and $\Phi(x, \mu)$, it follows that

$$\frac{d}{dx} W\{Z(x),\ \Phi(x, \mu)\} = (\mu_0 - \mu)Z(x)\Phi(x, \mu). \tag{6.2.5}$$

Using this and the equation satisfied by $Y_1(x)$, we conclude from (6.2.1) that

$$\begin{aligned} R'' - Q_1(x)R &= -\mu_0 R + 2Y_1'(x)Z(x)\Phi(x, \mu) + Y_1(x)[Z(x)\Phi(x, \mu)]' = \\ &= -\mu_0 R + 2[Y_1(x)Z(x)]'\Phi(x, \mu) + Y_1(x)W\{Z(x),\ \Phi(x, \mu)\}, \end{aligned}$$

i.e.,

$$R'' - Q_1(x)R + \mu R = 2[Y_1(x)Z(x)]'\Phi(x, \mu).$$

Subtraction of (6.2.3) from

$$\Phi'' - Q_1(x)\Phi + \mu\Phi = [Q(x) - Q_1(x)]\Phi(x, \mu),$$

gives for $\Psi = \Phi - R$ the identity

$$\Psi'' - Q_1(x)\Psi + \mu\Psi = \{Q(x) - Q_1(x) - 2[Y_1(x)Z(x)]'\}\Phi(x, \mu),$$

and this completes the proof of the lemma.

If $Y_1(x)$ is so chosen that $[Y_1(x)\,Z(x)]' = 0$, then the formula (6.2.1) yields the generalization of the transformation operator (6.1.8) to the matrix case.

However, since they will not be needed below, such operators will be considered no further.

LEMMA 6.2.2: *Let $Y(x)$ and $Z(x)$ be matrices satisfying*

$$Y''-Q(x)Y+\mu_0 Y = 0, \quad Z''-ZQ(x)+\mu_0 Z = 0 \qquad (a < x < b)$$

for which

$$W\{Z(x),\ Y(x)\}\equiv 0.$$

If $J(x)$ is an indefinite integral of $Z(x)Y(x)$ $[J'(x) = Z(x)Y(x)]$ possessing an inverse $J^{-1}(x)$ for all values of x in the interval (a, b), then in this interval the matrices

$$Y_1(x)=Y(x)J^{-1}(x), \quad Z_1(x)=J^{-1}(x)Z(x)$$

are solutions of

$$Y_1''-Q_1(x)Y_1+\mu_0 Y_1 = 0, \quad Z_1''-Z_1Q_1(x)+\mu_0 Z_1 = 0, \qquad (6.2.6)$$

such that

$$W\{Z_1(x),\ Y_1(x)\}\equiv 0,$$

where

$$Q_1(x) = Q(x)+\Delta(x)$$

and

$$\Delta(x) = Q_1(x)-Q(x) = -2[Y(x)Z_1(x)]' = -2[Y_1(x)Z(x)]'. \qquad (6.2.7)$$

Proof: For any non-singular differentiable matrix $A(x)$,

$$[A^{-1}(x)]' = -A^{-1}(x)A'(x)A^{-1}(x)$$

and therefore

$$[J^{-1}(x)]' = -J^{-1}(x)Z(x)Y(x)J^{-1}(x) = -Z_1(x)Y_1(x). \qquad (6.2.8)$$

Hence

$$Y_1'(x) = Y'(x)J^{-1}(x)-Y(x)Z_1(x)Y_1(x),$$
$$Z_1'(x) = J^{-1}(x)Z'(x)-Z_1(x)Y_1(x)Z(x),$$

from which it follows that

$$\begin{aligned} W\{Z_1(x),\ Y_1(x)\} &= Z_1(x)Y'(x)J^{-1}(x)-Z_1(x)Y(x)Z_1(x)Y_1(x)- \\ &\quad -J^{-1}(x)Z'(x)Y_1(x)+Z_1(x)Y_1(x)Z(x)Y_1(x) = \\ &= J^{-1}(x)[Z(x)Y'(x)-Z'(x)Y(x)]J^{-1}(x)-Z_1(x)[Y(x)J^{-1}(x)Z(x)- \\ &\quad -Y(x)J^{-1}(x)Z(x)]Y_1(x) = J^{-1}(x)W\{Z(x),\ Y(x)\}J^{-1}(x) \equiv 0, \end{aligned}$$

and thus one part of the lemma is proved.

Now consider

$$Y_1''(x) = Y''(x)J^{-1}(x)+2Y'(x)[J^{-1}(x)]'+Y(x)[J^{-1}(x)]''.$$

By (6.2.8),

$$Y_1''(x) = Y''(x)J^{-1}(x) - 2Y'(x)Z_1(x)Y_1(x) - \\ - Y(x)[Z_1'(x)Y_1(x) + Z_1(x)Y_1'(x)].$$

Since $W\{Z_1(x), Y_1(x)\} \equiv 0$ by the above,

$$Y_1''(x) = Y''(x)J^{-1}(x) - 2[Y(x)Z_1(x)]'Y_1(x).$$

Thus, using the equation defining $Y(x)$, we obtain

$$Y_1''(x) = [Q(x) - \mu_0 I]Y(x)J^{-1}(x) - 2[Y(x)Z_1(x)]'Y_1(x),$$

i.e.,

$$Y_1''(x) - \{Q(x) - 2[Y(x)Z_1(x)]'\}Y_1(x) + \mu_0 Y_1(x) = 0,$$

where

$$[Y(x)Z_1(x)]' = [Y(x)J^{-1}(x)Z(x)]' = [Y_1(x)Z(x)]'.$$

The derivation of (6.2.6) for $Z_1(x)$ is carried out similarly.

The above lemmas imply the following theorem.

THEOREM 6.2.1: *Let $Y(x)$ and $Z(x)$ be matrices satisfying*

$$Y'' - Q(x)Y + \mu_0 Y = 0, \quad Z'' - ZQ(x) + \mu_0 Z = 0 \qquad (a < x < b)$$

for which $W\{Z(x), Y(x)\} \equiv 0$. Let $J(x)$ be an indefinite integral of $Z(x)Y(x)$ $[J'(x) = Z(x)Y(x)]$ having an inverse in the interval $a < x < b$ and define

$$Y_1(x) = Y(x)J^{-1}(x), \quad Z_1(x) = J^{-1}(x)Z(x).$$

Then the transformation

$$\Psi(x, \mu) = \Phi(x, \mu) - Y_1(x)W\{Z(x), \Phi(x, \mu)\}(\mu_0 - \mu)^{-1} \qquad (6.2.9)$$

carries any solution $\Phi(x, \mu)$ $(\mu \neq \mu_0)$ of

$$\Phi'' - Q(x)\Phi + \mu\Phi = 0$$

into a solution $\Psi(x, \mu)$ of

$$\Psi'' - [Q(x) + \Delta(x)]\Psi + \mu\Psi = 0,$$

where

$$\Delta(x) = -2[Y(x)Z_1(x)]' = -2[Y_1(x)Z(x)]'.$$

The inverse transformation is given by

$$\Phi(x, \mu) = \Psi(x, \mu) + Y(x)W\{Z_1(x), \Psi(x, \mu)\}(\mu_0 - \mu)^{-1}, \qquad (6.2.10)$$

in which

$$Y(x) = -Y_1(x)J_1^{-1}(x), \quad Z(x) = -J_1^{-1}(x)Z_1(x),$$

and $J_1(x)$ *is that indefinite integral of* $Z_1(x)\,Y_1(x)$ *for which* $-J_1(x_0) = J^{-1}(x_0)$ *for some* x_0 *(and hence for all* $x \in (a, b)$*).*

Proof: The statement pertaining to the transformation (6.2.9) is an immediate consequence of formula (6.2.4) of Lemma 6.2.1 and the result (6.2.6) of Lemma 6.2.2. The expression (6.2.10) for the inverse transformation follows from the relation

$$Y(x)\,W\{Z_1(x),\ \Psi(x, \mu)\} = Y_1(x)\,W\{Z(x),\ \Phi(x, \mu)\},$$

which can be verified directly. Finally, from (6.2.8), it follows that $-J^{-1}(x)$ is an indefinite integral $J_1(x)$ of $Z_1(x)\,Y_1(x)$. Therefore, $Y(x) = Y_1(x)\,J(x) = -Y_1(x)\,J_1^{-1}(x)$ and $Z(x) = J(x)\,Z_1(x) = -J_1^{-1}(x)\,Z_1(x)$, where $J_1(x)$ is that indefinite integral of $Z_1(x)\,Y_1(x)$ which coincides with $-J^{-1}(x)$ for at least one value x_0 (and hence for all x).

The theorem is proved.

Remark. In the sequel, we shall need the special case of Theorem 6.2.1. in which $Q(x)$ is Hermitian and μ_0 is a real number. If $Y(x)$ is such that $W\{Y^*(x), Y(x)\} \equiv 0$, then we may take $Z(x)$ to be $\pm Y^*(x)$ and $J(x)$ to be a Hermitian (and non-singular) indefinite integral of $\pm Y^*(x)\,Y(x)$. In this connection, (6.2.9) becomes

$$\Psi(x, \mu) = \Phi(x, \mu) \mp Y_1(x)\,W\{Y^*(x),\ \Phi(x, \mu)\}\,(\mu_0-\mu)^{-1} \tag{6.2.11}$$

and $\Psi(x, \mu)$ satisfies the equation

$$\Psi'' - [Q(x) + \Delta(x)]\Psi + \mu\Psi = 0,$$

in which

$$Q(x) + \Delta(x) = Q(x) \mp 2[Y(x)J^{-1}(x)Y^*(x)]'$$

is also Hermitian.

§ 3. Transformation of Parseval's Equality

Consider the differential equation

$$U'' - Q(x)U + \mu U = 0 \qquad (a < x < b),$$

in which $Q(x)$ is a Hermitian matrix and μ a parameter.

We shall say that Parseval's equality holds for a family of solutions $U(x, \mu)$ of this equation if for $-\infty < \mu < \infty$, there exists a Hermitian matrix $\mathrm{P}(\mu)$, whose quadratic form is a non-decreasing function of μ, such that for any matrices $\Phi_1(x)$ and $\Phi_2(x)$ with elements in $L^2(a, b)$,

$$\int_a^b \Phi_1(x)\Phi_2^*(x)\,dx = \int_{-\infty}^{\infty} U(\Phi_1;\ \mu)\,dP(\mu)U^*(\Phi_2;\ \mu), \tag{6.3.1}$$

where

$$U(\Phi;\ \mu) = \underset{\substack{\alpha\to a\\ \beta\to b}}{\text{l.i.m.}} \int_\alpha^\beta \Phi(x)U(x,\ \mu)\,dx \tag{6.3.2}$$

(the last integral converges in the metric that is defined in the natural way in terms of $P(\mu)$).

Now, applying the transformation (6.2.11) to $U(x, \mu)$, we obtain a family of matrices $U_1(x, \mu)$ satisfying

$$U_1'' - [Q(x) + \Delta(x)]U_1 + \mu U_1 = 0,$$

in which

$$\Delta(x) = \mp 2[Y(x)J^{-1}(x)Y^*(x)]'.$$

We shall show that under certain conditions, a Parseval equality of a similar kind also holds for the family $U_1(x, \mu)$.

The transformation (6.2.11) is not of integral type, but it can be put into a form very close to that of an integral transformation. This is a consequence of the following simple lemma.

LEMMA 6.3.1: *If $Y^*(t)\ \Phi(t, \mu)$ is summable in the neighborhood of $t = a$, then* (6.2.11) *can be represented in the form*

$$\Psi(x,\ \mu) = \Phi(x,\ \mu) \mp \int_a^b H_a(x,\ t)\Phi(t,\ \mu)\,dt \mp Y_1(x)B_a(\mu), \tag{6.3.3}$$

where

$$H_a(x,\ t) = \begin{cases} Y_1(x)Y^*(t), & a < t \leq x < b \\ 0, & a < x < t < b, \end{cases} \tag{6.3.4}$$

$$B_a(\mu) = \lim_{x\to a} W\{Y^*(x),\ \Phi(x,\ \mu)\}\,(\mu_0 - \mu)^{-1}. \tag{6.3.5}$$

If $Y^*(t)\ \Phi(t, \mu)$ is summable in the neighborhood of $t = b$, then (6.2.11) *can be represented in the form*

$$\Psi(x,\ \mu) = \Phi(x,\ \mu) \pm \int_a^b H_b(x,\ t)\Phi(t,\ \mu)\,dt \mp Y_1(x)B_b(\mu), \tag{6.3.6}$$

where

$$H_b(x, t) = \begin{cases} 0, & a < t \leq x < b \\ Y_1(x)Y^*(t), & a < x < t < b, \end{cases}$$

$$B_b(\mu) = \lim_{x \to b} W\{Y^*(x),\ \Phi(x, \mu)\}(\mu_0 - \mu)^{-1}.$$

Proof: Suppose that the elements of the matrix $Y^*(t)\Phi(t, \mu)$ are summable in the neighborhood of $t = a$. Then setting $Z(x) = Y^*(x)$ in (6.2.5), we obtain

$$W\{Y^*(x),\ \Phi(x, \mu)\}(\mu_0 - \mu)^{-1} - W\{Y^*(\xi),\ \Phi(\xi, \mu)\}(\mu_0 - \mu)^{-1} = \\ = \int_\xi^x Y^*(t)\Phi(t, \mu)\, dt.$$

Hence, when $\xi \to a$,

$$W\{Y^*(x),\ \Phi(x, \mu)\}(\mu_0 - \mu)^{-1} = \int_a^x Y^*(t)\Phi(t, \mu)\, dt + B_a(\mu),$$

where

$$B_a(\mu) = \lim_{\xi \to a} W\{Y^*(\xi),\ \Phi(\xi, \mu)\}(\mu_0 - \mu)^{-1}.$$

Therefore, (6.2.11) is expressible as

$$\Psi(x, \mu) = \Phi(x, \mu) \mp \int_a^x Y_1(x)Y^*(t)\Phi(t, \mu)\, dt \mp Y_1(x)B_a(\mu),$$

and this is equivalent to (6.3.3).

The proof of formula (6.3.6) for the case where $Y^*(t)\Phi(t, \mu)$ is summable in the neighborhood of $t = b$ is completely analogous.

Remark. The expressions (6.3.3) and (6.3.6) have been derived under the assumption that $\mu \neq \mu_0$. However if in addition to the hypothesis of the lemma, the Wronskian $W\{Y^*(x), \Phi(x, \mu_0)\}$ vanishes (the Wronskian is clearly independent of x), then

$$\Psi(x, \mu_0) = \Phi(x, \mu_0) \mp \int_a^b H_a(x, t)\Phi(t, \mu_0)\, dt$$

$$\left(\text{or} \quad \Psi(x, \mu_0) = \Phi(x, \mu_0) \pm \int_a^b H_b(x, t)\Phi(t, \mu_0)\, dt\right)$$

is also a solution of the transformed equation when $\mu = \mu_0$. This is shown most simply by direct verification.

THEOREM 6.3.1: *Let Parseval's equality* (6.3.1) *hold for the family of matrices* $U(x, \mu)$, *and suppose that this family is transformed into* $U_1(x, \mu)$ *by means of* (6.3.3) (*or* (6.3.6)), *in which the elements of* $Y(t)$ *are square summable in the neighborhood of* $t = a$ (*or* $t = b$) *and* $B_a(\mu)$ (*or* $B_b(\mu)$) *satisfies the conditions:*

(1) $$\int_{-\infty}^{\infty} B_a(\mu)\, d\mathrm{P}(\mu) B_a^*(\mu) \qquad \left(\int_{-\infty}^{\infty} B_b(\mu)\, d\mathrm{P}(\mu) B_b^*(\mu)\right) \quad \text{exists};$$

(2) for all matrices $\Psi(x)$ *with elements in* $L^2(a, b)$,

$$\int_{-\infty}^{\infty} U(\Psi;\ \mu)\, d\mathrm{P}(\mu) B_a^*(\mu) = 0 \qquad \left(\int_{-\infty}^{\infty} U(\Psi;\ \mu)\, d\mathrm{P}(\mu) B_b^*(\mu) = 0\right);$$

If $\Phi(x)$ *is any matrix with elements in* $L^2(a, b)$ *that vanish in the neighborhood of* $x = a$ *and* $x = b$, *then*

$$\int_a^b \Phi(x)\Phi^*(x)\, dx = \int_{-\infty}^{\infty} U_1(\Phi;\ \mu)\, d\mathrm{P}(\mu) U_1^*(\Phi;\ \mu) + \\ + Y_1(\Phi)[\pm J(a) - \boldsymbol{B}(a)] Y_1^*(\Phi) \tag{6.3.7}$$

(*or*

$$\int_a^b \Phi(x)\Phi^*(x)\, dx = \int_{-\infty}^{\infty} U_1(\Phi;\ \mu)\, d\mathrm{P}(\mu) U_1^*(\Phi;\ \mu) - \\ - Y_1(\Phi)\, [\pm J(b) + \boldsymbol{B}(b)] Y_1^*(\Phi)\Big), \tag{6.3.8}$$

with

$$U_1(\Phi;\ \mu) = \int_a^b \Phi(x) U_1(x,\ \mu)\, dx, \qquad Y_1(\Phi) = \int_a^b \Phi(x) Y_1(x)\, dx;$$

$$\boldsymbol{B}(a) = \int_{-\infty}^{\infty} B_a(\mu)\, d\mathrm{P}(\mu) B_a^*(\mu), \qquad \boldsymbol{B}(b) = \int_{-\infty}^{\infty} B_b(\mu)\, d\mathrm{P}(\mu) B_b^*(\mu).$$

Proof: Suppose that $U(x, \mu)$ is transformed into $U_1(x, \mu)$ by means of (6.3.3) and that the elements of $Y(x)$ are square summable in the neighborhood of $x = a$. Let us premultiply both sides of (6.3.3) by some arbitrary matrix $\Phi(x)$ with elements in $L^2(a, b)$ vanishing in the neighborhood of $x = a$ and $x = b$, and let us integrate the result over the interval (a, b). This yields

$$U_1(\Phi;\ \mu) = U(\Phi;\ \mu) \mp \\ \mp \int_a^b \Phi(x)\, dx \int_a^b H_a(x,\ t) U(t,\ \mu)\, dt \mp Y_1(\Phi) B_a(\mu).$$

Interchanging the order of integration and defining

$$\Phi_H(x) = \int_a^b \Phi(\xi) H_a(\xi, x)\, d\xi,$$

we then have

$$U_1(\Phi;\ \mu) = U(\Phi;\ \mu) \mp U(\Phi_H;\ \mu) \mp Y_1(\Phi) B_a(\mu).$$

Moreover, the elements of $\Phi_H(x)$ can easily be shown to belong to $L^2(a, b)$. Therefore, applying Parseval's equality (6.3.1) and the conditions (*1*) and (*2*) on $B_a(\mu)$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} U_1(\Phi;\ \mu)\, dP(\mu) U_1^*(\Phi;\ \mu) &= \int_a^b \Phi(x)\Phi^*(x)\, dx \mp \\ \mp \int_a^b \Phi(x)\Phi_H^*(x)\, dx \mp \int_a^b \Phi_H(x)\Phi^*(x)\, dx &+ \int_a^b \Phi_H(x)\Phi_H^*(x)\, dx + \\ &+ Y_1(\Phi) B(a) Y_1^*(\Phi). \end{aligned} \tag{6.3.9}$$

We next find that

$$\begin{aligned} \mp \int_a^b \Phi(x)\Phi_H^*(x)\, dx \mp \int_a^b \Phi_H(x)\Phi^*(x)\, dx + \int_a^b \Phi_H(x)\Phi_H^*(x) dx &= \\ = \int_a^b \int_a^b \Phi(x) \left\{ \mp H_a^*(\xi, x) \mp H_a(x, \xi) + \right. & \\ \left. + \int_a^b H_a(x, t) H_a^*(\xi, t)\, dt \right\} \Phi^*(\xi)\, d\xi\, dx & \end{aligned}$$

and using (6.3.4) for $x \leq \xi$, that

$$\begin{aligned} \mp H_a^*(\xi, x) \mp H_a(x, \xi) + \int_a^b H_a(x, t) H_a^*(\xi, t)\, dt &= \\ = \mp Y_1(x) Y^*(\xi) + Y_1(x) \int_a^{\xi} Y^*(t) Y(t)\, dt \cdot Y_1^*(\xi) &= \\ = Y_1(x) \{\mp Y^*(\xi) \pm [J(\xi) - J(a)] Y_1^*(\xi)\} &= \mp Y_1(x) J(a) Y_1^*(\xi), \end{aligned}$$

since by definition $Y^*(\xi) = J(\xi) Y_1^*(\xi)$.

A simple computation shows that this result is also true for $x < \xi$.

Therefore,

$$\mp \int_a^b \Phi(x)\Phi_H^*(x)\,dx \mp \int_a^b \Phi_H(x)\Phi^*(x)\,dx + \int_a^b \Phi_H(x)\Phi_H^*(x)\,dx =$$

$$= \mp \int_a^b \int_a^b \Phi(x)Y_1(x)J(a)Y_1^*(\xi)\Phi^*(\xi)\,d\xi\,dx =$$

$$= \mp Y_1(\Phi)J(a)Y_1^*(\Phi),$$

and from this and (6.3.9), we have

$$\int_{-\infty}^{\infty} U_1(\Phi;\ \mu)\,d\mathrm{P}(\mu)U_1^*(\Phi;\ \mu) =$$

$$= \int_a^b \Phi(x)\Phi^*(x)\,dx - Y_1(\Phi)\,[\pm J(a) - \boldsymbol{B}(a)]Y_1^*(\Phi),$$

This is equivalent to (6.3.7).

The expression (6.3.8) can be derived in a similar way; in this case the transformation has the form (6.36), and the elements of $Y(x)$ are supposed to be square summable in the neighborhood of $x = b$.

CHAPTER VII

Spectral Analysis of the Boundary-Value Problem with Singularities

§ 1. Statement of the Problem. Notation

The general scheme for treating a singular boundary-value problem described in §1 of the preceding chapter leads in each specific instance to a more or less painstaking analysis. Therefore, we shall confine ourselves to the detailed consideration of one such problem which is encountered in the theory of the deuteron.

This boundary-value problem consists of the matrix differential equation

$$Y''-[V(x)+6x^{-2}P]Y+\lambda^2 Y = 0 \qquad (0 < x < \infty) \qquad (7.1.1)$$

and the boundary condition

$$Y(0,\ \lambda)=0,$$

where $V(x)$ is a second order Hermitian matrix and P a projection of rank 1.

Concerning $V(x)$, which we agree to call the *potential matrix*, we make the assumption that for some ε $(0 < \varepsilon < 1)$

$$\int_0^\infty x^{1+\theta}\,|V(x)|\,dx<\infty \qquad (|\,\theta\,| <\varepsilon). \qquad (7.1.1)$$

Thus, the singularities of the problem in question are contained only in the term $6x^{-2}P$.

Without loss of generality, we may also take

$$P=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (7.1.2)$$

since the general case can be reduced to this simpler formulation.

The boundary-value problem described by equation (7.1.0) (with $V(x)$ satisfying the inequality (7.1.1)) and the boundary condition $Y(0, \lambda) = 0$ will be denoted in the sequel by (7.1.0)—(0). In the course of our investigation, we shall encounter problems characterized by other differential equations but the same boundary condition $Y(0, \lambda) = 0$, and we shall agree to use the analogous notation (. . .)—(0), in which the number of the corresponding differential equation will be indicated in the first pair of parentheses.

§ 2. Particular Solutions

In the later presentation, certain particular solutions of

$$Y'' - [V(x) + 6x^{-2}P]Y = 0. \tag{7.2.1}$$

will be needed.

Theorem 7.2.1: *If for some* $\delta \geq 0$ $(\delta < 2)$ $V(x) = \|v_{ij}\|_1^2$ *satisfies the condition*

$$\int^{\infty} x^{1+\delta} \,|\, V(x) \,|\, dx < \infty, \tag{7.2.2}$$

then (7.2.1) *has a fundamental system of solutions* $Y_\infty^{(1)}(x)$ *and* $Y_\infty^{(2)}(x)$ *such that*

$$Y_\infty^{(1)}(x) = \begin{pmatrix} 1 + o(x^{-\delta}) & o(x^{-2-\delta}) \\ o(x^{-\delta}) & x^{-2} + o(x^{-2-\delta}) \end{pmatrix} \qquad (x \to \infty),$$

$$Y_\infty^{(2)}(x) = \begin{pmatrix} x + o(x^{1-\delta}) & o(x^{3-\delta}) \\ o(x^{1-\delta}) & x^{3} + o(x^{3-\delta}) \end{pmatrix} \qquad (x \to \infty).$$

These relations may be differentiated to obtain asymptotic relations for the derivatives.

Proof: Assuming that $0 < a \leq x < \infty$, consider the system of integral equations

$$y_{11}^{(1)}(x) = 1 + \int_x^\infty (t - x)\,[v_{11}(t) y_{11}^{(1)}(t) + v_{12}(t) y_{21}^{(1)}(t)]\, dt,$$

$$y_{21}^{(1)}(x) = -\frac{x^3}{5} \int_x^\infty t^{-2}[v_{21}(t) y_{11}^{(1)}(t) + v_{22}(t) y_{21}^{(1)}(t)]\, dt -$$

$$- \frac{x^{-2}}{5} \int_a^x t^3 [v_{21}(t) y_{11}^{(1)}(t) + v_{22}(t) y_{21}^{(1)}(t)]\, dt. \tag{7.2.3}$$

This system can be expressed in the form

$$\eta(x) = \eta_0 + \int_a^\infty A(x,\, t)\eta(t)\, dt, \tag{7.2.4}$$

where

$$\eta(x) = \begin{pmatrix} y_{11}^{(1)}(x) \\ y_{21}^{(1)}(x) \end{pmatrix}, \qquad \eta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$A(x,\, t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ -\frac{1}{5} x^{-2} t^3 v_{21}(t) & -\frac{1}{5} x^{-2} t^3 v_{22}(t) \end{pmatrix}, & a \le t \le x \\ \begin{pmatrix} (t-x)v_{11}(t) & (t-x)v_{12}(t) \\ -\frac{1}{5} x^3 t^{-2} v_{21}(t) & -\frac{1}{5} x^3 t^{-2} v_{22}(t) \end{pmatrix}, & x \le t < \infty. \end{cases}$$

From this expression for the kernel $A(x, t)$ and from (7.2.2), it follows that

$$\int_a^\infty |\, A(x,\, t)\,|\, dt \le q < 1$$

for a sufficiently large and for any x in the interval $a \le x < \infty$. Therefore using the method of successive approximations, we can show that (7.2.4) has a unique bounded solution in the interval $a \le x < \infty$ for a sufficiently large. In this connection,

$$|\,\eta(x)\,| \le (1-q)^{-1}.$$

By means of this estimate and (7.2.2), we then find from (7.2.3) that

$$y_{11}^{(1)}(x) = 1 + o(x^{-\delta}), \qquad y_{21}^{(1)}(x) = o(x^{-\delta})$$

as $x \to \infty$.

From the equations obtained by differentiating (7.2.3), we conclude in the same way that as $x \to \infty$

$$[y_{11}^{(1)}(x)]' = o(x^{-1-\delta}), \qquad [y_{21}^{(1)}(x)]' = o(x^{-1-\delta}).$$

Finally, by the differentiation of the relations in (7.2.3), we can verify directly that $\eta(x)$ is a solution of the system (7.2.1). The vector $\eta(x)$ forms the first column of the matrix $Y_\infty^{(1)}(x)$.

Now consider the equation

$$\zeta(x) = \zeta_0(x) + \int_a^\infty A(x,\, t)\zeta(t)\, dt, \tag{7.2.5}$$

in which

$$\zeta(x)=\begin{pmatrix} y_{12}^{(1)}(x) \\ y_{22}^{(1)}(x) \end{pmatrix}, \qquad \zeta_0(x)=\begin{pmatrix} 0 \\ x^{-2} \end{pmatrix},$$

and the kernel $A(x, t)$ is the same as that of (7.2.4). Any solution $\zeta(x)$ of the integral equation (7.2.5) is also a solution of the system (7.2.1). Transcribing (7.2.5) into the form

$$x^2\zeta(x) = x^2\zeta_0(x)+\int_a^\infty x^2t^{-2}A(x,\ t)t^2\zeta(t)\,dt \tag{7.2.6}$$

and noting that by (7.2.2),

$$\int_a^\infty x^2t^{-2}\,|\,A(x,\ t)\,|\,dt \leq q < 1 \qquad (a \leq x < \infty)$$

for a sufficiently large, we conclude that (7.2.6) has a unique solution $x^2\zeta(x)$ bounded at infinity, by the method of successive approximations. Using (7.2.2), we then find from (7.2.6) that $x^2\zeta(x) = x^2\zeta_0(x) + o(x^{-\delta})$ as $x \to \infty$; moreover, this asymptotic relation may be differentiated. Thus, the equation (7.2.5), and therefore (7.2.1) as well, has a vector solution $\zeta(x)$, with components having the following asymptotic behavior at infinity:

$$y_{12}^{(1)}(x) = o(x^{-2-\delta}), \qquad [y_{12}^{(1)}(x)]' = o(x^{-3-\delta});$$

$$y_{22}^{(1)}(x) = x^{-2}+o(x^{-2-\delta}), \qquad [y_{22}^{(1)}(x)]' = -2x^{-3}+o(x^{-3-\delta}).$$

The vector $\zeta(x)$ yields the second column of $Y_\infty^{(1)}(x)$, and thus the existence of the matrix solution $Y_\infty^{(1)}(x)$ of (7.2.1) is proved.

The existence of the matrix solution $Y_\infty^{(2)}(x)$ is proved similarly. Its columns are the respective solutions of the following two systems of integral equations:

$$y_{11}^{(2)} = x+\int_a^x (x-t)\,[v_{11}(t)y_{11}^{(2)}(t)+v_{12}(t)y_{21}^{(2)}(t)]\,dt,$$

$$y_{21}^{(2)}(x) = -\frac{x^{-2}}{5}\int_a^x t^3[v_{21}(t)y_{11}^{(2)}(t)+v_{22}(t)y_{21}^{(2)}(t)]\,dt -$$

$$-\frac{x^3}{5}\int_x^\infty t^{-2}[v_{21}(t)y_{11}^{(2)}(t)+v_{22}(t)y_{21}^{(2)}(t)]\,dt$$

and

$$\begin{aligned} y_{12}^{(2)}(x) &= \int_a^x (x-t)\,[v_{11}(t)y_{12}^{(2)}(t)+v_{12}(t)y_{22}^{(2)}(t)]\,dt,\\ y_{22}^{(2)}(x) &= x^3-\frac{x^{-2}}{5}\int_a^x t^3[v_{21}(t)y_{12}^{(2)}(t)+v_{22}(t)y_{22}^{(2)}(t)]\,dt-\\ &\quad -\frac{x^3}{5}\int_x^\infty t^{-2}[v_{21}(t)y_{12}^{(2)}(t)+v_{22}(t)y_{22}^{(2)}(t)]\,dt. \end{aligned}$$

Solutions to these systems can be shown to exist in the interval $a \leq x < \infty$ when a is sufficiently large.

THEOREM 7.2.2: *If for some* $\delta \geq 0$ $(\delta < 2)$ $V(x)$ *is a matrix for which*

$$\int_0 x^{1-\delta}\,|V(x)|\,dx<\infty, \tag{7.2.7}$$

then the equation (7.2.1) *has a fundamental system of solutions* $Y_0^{(1)}(x)$ *and* $Y_0^{(1)}(x)$ *such that*

$$\begin{aligned} Y_0^{(1)}(x) &= \begin{pmatrix} 1+o(x^\delta) & o(x^{-2+\delta}) \\ o(x^\delta) & x^{-2}+o(x^{-2+\delta}) \end{pmatrix} \qquad (x\to 0),\\ Y_0^{(2)}(x) &= \begin{pmatrix} x+o(x^{1+\delta}) & o(x^{3+\delta}) \\ o(x^{1+\delta}) & x^3+o(x^{3+\delta}) \end{pmatrix} \qquad (x\to 0). \end{aligned}$$

These relations may be differentiated to obtain asymptotic relations for the derivatives.

The proof is similar to that of the preceding theorem, and we shall therefore merely give it in outlined form.

Consider the following four systems of integral equations:

$$\left.\begin{aligned} y_{11}^{(1)}(x) &= 1-\int_0^x t[v_{11}(t)y_{11}^{(1)}(t)+v_{12}(t)y_{21}^{(1)}(t)]\,dt-\\ &\quad -x\int_x^a [v_{11}(t)y_{11}^{(1)}(t)+v_{12}(t)y_{21}^{(1)}(t)]\,dt,\\ y_{21}^{(1)}(x) &= -\frac{x^{-2}}{5}\int_0^x t^3[v_{21}(t)y_{11}^{(1)}(t)+v_{22}(t)y_{21}^{(1)}(t)]\,dt-\\ &\quad -\frac{x^3}{5}\int_x^a t^{-2}[v_{21}(t)y_{11}^{(1)}(t)+v_{22}(t)y_{21}^{(1)}(t)]\,dt; \end{aligned}\quad (0\leq x\leq a)\right\}$$

$$\left.\begin{aligned} y^{(1)}_{12}(x) &= \int_x^a (t-x)\,[v_{11}(t)y^{(1)}_{12}(t)+v_{12}(t)y^{(1)}_{22}(t)]\,dt,\\ y^{(1)}_{22}(x) &= x^{-2}-\frac{x^{-2}}{5}\int_0^x t^3[v_{21}(t)y^{(1)}_{12}(t)+v_{22}(t)y^{(1)}_{22}(t)]\,dt- \\ &\quad -\frac{x^3}{5}\int_x^a t^{-2}[v_{21}(t)y^{(1)}_{12}(t)+v_{22}(t)y^{(1)}_{22}(t)]\,dt; \end{aligned}\qquad (0 \leq x \leq a)\right\}$$

$$\left.\begin{aligned} y^{(2)}_{11}(x) &= x+\int_0^x (x-t)\,[v_{11}(t)y^{(2)}_{11}(t)+v_{12}(t)y^{(2)}_{21}(t)]\,dt,\\ y^{(2)}_{21}(x) &= -\frac{x^{-2}}{5}\int_0^x t^3[v_{21}(t)y^{(2)}_{11}(t)+v_{22}(t)y^{(2)}_{21}(t)]\,dt- \\ &\quad -\frac{x^3}{5}\int_x^a t^{-2}[v_{21}(t)y^{(2)}_{11}(t)+v_{22}(t)y^{(2)}_{21}(t)]\,dt; \end{aligned}\qquad (0 \leq x \leq a)\right\}$$

$$\left.\begin{aligned} y^{(2)}_{12}(x) &= \int_0^x (x-t)\,[v_{11}(t)y^{(2)}_{12}(t)+v_{12}(t)y^{(2)}_{22}(t)]\,dt,\\ y^{(2)}_{22}(x) &= x^3+ \\ &\quad +\frac{1}{5}\int_0^x (x^3t^{-2}-x^{-2}t^3)\,[v_{21}(t)y^{(2)}_{12}(t)+v_{22}(t)y^{(2)}_{22}(t)]\,dt. \end{aligned}\qquad (\leq x < \infty)\right\}$$

Any solution of each of these systems is simultaneously a solution of the system(7.2.1).

By the use of (7.2.7), it is possible to show by the method of successive approximations that if a is sufficiently small, then each of these systems has a unique solution. The solutions of the first two systems comprise the columns of $Y^{(1)}_0(x)$ and those of the last two systems the columns of $Y^{(2)}_0(x)$. Furthermore, $Y^{(1)}_0(x)$ and $Y^{(2)}_0(x)$ will have the asymptotic behavior at $x = 0$ stated in the theorem.

REMARK. If $V(x)$ satisfies the condition (7.2.7), then obviously so will the matrix $V(x)-\lambda^2I$ for any value of λ.

Therefore, (7.2.7) assures the existence of a fundamental system of matrix solutions of

$$Y'' - [V(x) + 6x^{-2}P]Y + \lambda^2 Y = 0$$

for any value of λ, and these will have the same asymptotic behavior at $x = 0$ as do $Y_0^{(1)}(x)$ and $Y_0^{(2)}(x)$.

§ 3. The First Transformation

1. Consider the matrix differential equation

$$Y'' - [V(x) + 6x^{-2}P]Y + \lambda^2 Y = 0, \qquad 0 < x < \infty,$$

where $V(x) = \| v_{ij} \|_1^2$ is a Hermitian matrix satisfying (7.1.1) and P is the projection (7.1.2).

Let us denote by $\Phi^{(0)}(x, \lambda)$ the solutions of this equation which at $x = 0$ have the asymptotic behavior

$$\Phi^{(0)}(x, \lambda) = \begin{pmatrix} x + o(x^{1+\theta}) & o(x^{3+\theta}) \\ o(x^{1+\theta}) & x^3 + o(x^{3+\theta}) \end{pmatrix} \quad (\theta < \varepsilon). \qquad (7.3.1)$$

The existence of such solutions follows from the remark of Theorem 7.2.2. The same remark implies that any solution of (7.1.0) vanishing at $x = 0$ is of the form $\Phi^{(0)}(x, \lambda)\, C(\lambda)$, where $C(\lambda)$ is a matrix not depending on x.

In this and the next two sections, it will be shown that with the aid of the transformations of § 2, Chapter VI the matrices $\Phi^{(0)}(x, \lambda)$ can be converted into solutions of equations without singularities also vanishing at $x = 0$. This will permit the reduction of the problem (7.1.0)—(0) to some non-singular problem investigated in detail in the first part of the book.

We begin by removing the singularity at infinity. By Theorem 7.2.1, the equation (7.1.0) for $\lambda = 0$ has a solution $Y(x) = Y_\infty^{(1)}(x)P$ with the asymptotic behavior

$$Y(x) = \begin{pmatrix} 0 & o(x^{-2-\theta}) \\ 0 & x^{-2} + o(x^{-2-\theta}) \end{pmatrix} = [x^{-2}I + o(x^{-2-\theta})]P \quad (x \to \infty), \quad (7.3.2)$$

for any $\theta < \varepsilon$, and this relation may be differentiated.

We recall that the symbol $o\{\varphi(x)\}$ occuring in any matrix equation is to denote a matrix with elements of the order $o\{\varphi(x)\}$.

From (7.3.2), it follows that $W\{Y^*(x), Y(x)\} \equiv 0$ (since this holds when $x \to \infty$), and that the matrix

$$Y_1(x) = Y(x)\left[I-P-\int_x^\infty Y^*(t)Y(t)\,dt\right]^{-1} \tag{7.3.3}$$

exists for all positive values of x.

Therefore, if $\Phi(x, \lambda)$ ($\lambda \neq 0$) is any solution of (7.1.0), then by Theorem 6.2.1 and its remark, the matrix $\Phi_1(x, \lambda)$ defined by

$$\Phi_1(x, \lambda) = \Phi(x, \lambda)+Y_1(x)W\{Y^*(x), \Phi(x, \lambda)\}\lambda^{-2} \tag{7.3.4}$$

satisfies

$$Y''-V_1(x)Y+\lambda^2 Y = 0, \qquad 0 < x < \infty, \tag{7.3.A}$$

where

$$V_1(x) = V(x)+6x^{-2}P-2[Y_1(x)Y^*(x)]'. \tag{7.3.5}$$

The application of (7.3.2) then yields for any $\theta < \varepsilon$

$$\left[I-P-\int_x^\infty Y^*(t)Y(t)\,dt\right]^{-1} = I-P-3x^3P+Po(x^{3-\theta})P \quad (x \to \infty),$$

$$Y_1(x) = [-3xI+o(x^{1-\theta})]P \qquad (x \to \infty), \tag{7.3.6}$$

and these relations may be differentiated. Therefore,

$$2[Y_1(x)Y^*(x)]' = 6x^{-2}P+o(x^{-2-\theta}) \qquad (x \to \infty)$$

and by (7.3.5), $V_1(x) = V(x)+o(x^{-2-\theta})$. This means that for any $\theta < \varepsilon$, $V_1(x)$ satisfies the condition

$$\int^\infty x^{1+\theta}\,|\,V_1(x)\,|\,dx<\infty. \tag{7.3.7}$$

In order to determine the asymptotic behavior of $V_1(x)$ at zero we make use of the fundamental system of solutions $Y_0^{(1)}(x)$ and $Y_0^{(2)}(x)$ whose existence was proved in Theorem 7.2.2. We can write

$$Y(x) = Y_0^{(1)}(x)AP+Y_0^{(2)}(x)BP,$$

where $A = \|\,a_{ij}\,\|_1^2$ and $B = \|\,b_{ij}\,\|_1^2$ are certain constant matrices. Hence, for any $\theta < \varepsilon$,

$$Y(x) = \begin{pmatrix} 1+o(x^\theta) & o(x^{-2+\theta}) \\ o(x^\theta) & x^{-2}+o(x^{-2+\theta}) \end{pmatrix}\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}+ \\ +\begin{pmatrix} 0 & b_{12}x+o(x^{1+\theta}) \\ 0 & o(x^{1+\theta}) \end{pmatrix} \qquad (x \to 0). \tag{7.3.8}$$

Suppose first that $a_{22} \neq 0$. Then as $x \to 0$,

$$Y(x) = [a_{22}x^{-2}I + o(x^{-2+\theta})]P, \tag{7.3.9}$$

$$\left[I - P - \int_x^\infty Y^*(t)Y(t)\,dt\right]^{-1} = \tag{7.3.9'}$$

$$= I - P - 3\,|\,a_{22}\,|^{-2}x^3P + Po(x^{3+\theta})P,$$

$$Y_1(x) = [-3a_{22}^{-1}xI + o(x^{1+\theta})]P. \tag{7.3.10}$$

These relations, as one can easily verify, may be differentiated. Therefore

$$2[Y_1(x)Y^*(x)]' = 6x^{-2}P + o(x^{-2+\theta}) \qquad (x \to 0)$$

and by (7.3.5), $V_1(x) = V(x) + o(x^{-2+\theta})$ as $x \to 0$. Thus for the case in question ($a_{22} \neq 0$),

$$\int_0 x^{1-\theta}\,|\,V_1(x)\,|\,dx < \infty \tag{7.3.11}$$

for any $\theta < \varepsilon$.

Now let $a_{22} = 0$ but $a_{12} \neq 0$. In that case, for any $\theta < \varepsilon$, as $x \to 0$ (see (7.3.8))

$$Y(x) = \begin{pmatrix} 0 & a_{12} + o(x^\theta) \\ 0 & o(x^\theta), \end{pmatrix}, \tag{7.3.12}$$

$$\left[I - P - \int_x^\infty Y^*(t)Y(t)\,dt\right]^{-1} = I - P - m^2P + Po(x)P, \tag{7.3.12'}$$

in which

$$m^{-2}P = \int_0^\infty Y^*(t)Y(t)\,dt. \tag{7.3.13}$$

Hence, for any $\theta < \varepsilon$

$$Y_1(x) = \begin{pmatrix} 0 & -a_{12}m^2 + o(x^\theta) \\ 0 & o(x^\theta) \end{pmatrix} \quad (x \to 0), \tag{7.3.14}$$

and again this relation may be differentiated. Therefore when $a_{22} = 0$ and $a_{12} \neq 0$, we have

$$2[Y_1(x)Y^*(x)]' = o(x^{-1+\theta}), \quad V_1(x) = V(x) + 6x^{-2}P + o(x^{-1+\theta})$$

as $x \to 0$ so that

$$\int_0 x^{1-\theta}\,|\,V_1(x) - 6x^{-2}P\,|\,dx < \infty \qquad (\theta < \varepsilon). \tag{7.3.15}$$

Finally, if $a_{22} = a_{12} = 0$, then by (7.3.8)

$$Y(x) = \begin{pmatrix} 0 & b_{12}x + o(x^{1+\theta}) \\ 0 & o(x^{1+\theta}) \end{pmatrix} \qquad (x \to 0), \tag{7.3.16}$$

and it is easy to verify that this again leads to the inequality (7.3.15).

Thus we have proved the following

LEMMA 7.3.1: *Any matrix $\Phi_1(x, \lambda)$ obtained from a solution $\Phi(x, \lambda)$ of* (7.1.0) *by means of the transformation* (7.3.4) *is a solution of*

$$Y'' - V_1(x)Y + \lambda^2 Y = 0, \qquad 0 < x < \infty, \tag{7.3.A}$$

in which $V_1(x)$ has the following properties:

(1) $$\int^{\infty} x^{1+\theta} \,|\, V_1(x) \,|\, dx < \infty \qquad (\theta < \varepsilon);$$

(2) if $a_{22} \neq 0$, i.e, $Y(x) \to \infty$ *as* $x \to 0$, *then*

$$\int_0 x^{1-\theta} \,|\, V_1(x) \,|\, dx < \infty \qquad (\theta < \varepsilon);$$

(3) if $a_{22} = 0$, i.e., $Y(x)$ is bounded at $x = 0$, then

$$\int_0 x^{1-\theta} \,|\, V_1(x) - 6x^{-2}P \,|\, dx < \infty \qquad (\theta < \varepsilon).$$

The other properties of the transformation (7.3.4) that will be needed are contained in the following there lemmas.

LEMMA 7.3.2: *Any matrix $\Phi_1^{(0)}(x, \lambda)$ obtained from a solution $\Phi^{(0)}(x, \lambda)$ of* (7.1.0) *(see the formula* (7.3.1)) *by means of the transformation* (7.3.4) *has the following asymptotic behavior at $x = 0$:*

$$\begin{aligned} \Phi_1^{(0)}(x, \lambda) &= O(x), && \text{if} \quad a_{22} \neq 0; \\ \Phi_1^{(0)}(x, \lambda) &= \Phi^{(0)}(x, \lambda) + O(x^3), && \text{if} \quad a_{22} = a_{12} = 0; \\ \Phi_1^{(0)}(x, \lambda) &= \Phi^{(0)}(x, \lambda) + Y_1(x)B_0(\lambda) + O(x^2), && \text{if} \quad a_{22} = 0,\ a_{12} \neq 0, \end{aligned}$$

where

$$B_0(\lambda) = \begin{pmatrix} 0 & 0 \\ \bar{a}_{12}\lambda^{-2} & 0 \end{pmatrix}; \qquad Y_1(x) = \begin{pmatrix} 0 & -a_{12}m^2 + o(x^\theta) \\ 0 & o(x^\theta) \end{pmatrix} \quad (x \to 0).$$

These asymptotic relations may be differentiated.

Proof: Let $a_{22} \neq 0$. Applying Lemma 6.3.1, we can write (7.3.4) in the form

$$\Phi_1^{(0)}(x, \lambda) = \Phi^{(0)}(x, \lambda) + Y_1(x) \left[\int_x^1 Y^*(t)\Phi^{(0)}(t, \lambda)\, dt + B_1(\lambda) \right],$$

where

$$B_1(\lambda) = W\{Y^*(x),\ \Phi^{(0)}(x,\lambda)\}\,|_{x=1}\lambda^{-2}.$$

By (7.3.1), (7.3.9) and (7.3.10), we find from this that $\Phi_1^{(0)}(x,\lambda) = O(x)$ as $x \to 0$.

If $a_{22} = a_{12} = 0$, then by (7.3.1) and (7.3.16)

$$\lim_{x\to 0} W\{Y^*(x),\ \Phi^{(0)}(x,\lambda)\} = 0,$$

and the transformation (7.3.4) becomes

$$\Phi_1^{(0)}(x,\lambda) = \Phi^{(0)}(x,\lambda) - Y_1(x)\int_0^x Y^*(t)\Phi^{(0)}(t,\lambda)\,dt.$$

Thus, by (7.3.16) and (7.3.3), $\Phi_1^{(0)}(x,\lambda) = \Phi^{(0)}(x,\lambda) + O(x^3)$ as $x \to 0$.

Finally, let $a_{22} = 0$, but $a_{12} \neq 0$. Then by (7.3.1) and (7.3.12),

$$\lim_{x\to 0} W\{Y^*(x),\ \Phi^{(0)}(x,\lambda)\}\,\lambda^{-2} = \lambda^{-2}Y^*(0)\Phi^{(0)\prime}(0,\lambda) = B_0(\lambda),$$

where

$$B_0(\lambda) = \begin{pmatrix} 0 & 0 \\ \bar{a}_{12}\lambda^{-2} & 0 \end{pmatrix}.$$

Applying Lemma 6.3.1, we obtain

$$\Phi_1^{(0)}(x,\lambda) = \Phi^{(0)}(x,\lambda) - Y_1(x)\int_0^x Y^*(t)\Phi^{(0)}(t,\lambda)\,dt + Y_1(x)B_0(\lambda).$$

This expression in conjunction with (7.3.1), (7.3.12), and (7.3.14) then yields

$$\Phi_1^{(0)}(x,\lambda) = \Phi^{(0)}(x,\lambda) + Y_1(x)B_0(\lambda) + O(x^2)$$

as $x \to 0$, q.e.d.

LEMMA 7.3.3: *The transformation inverse to* (7.3.4) *is given by*

$$\Phi(x,\lambda) = \Phi_1(x,\lambda) - Y(x)\,W\{Y_1^*(x),\ \Phi_1(x,\lambda)\}\,\lambda^{-2}, \quad (7.3.17)$$

where

$$Y(x) = Y_1(x)\left[I - P - \int_0^x Y_1^*(t)Y_1(t)\,dt\right]^{-1}, \quad \text{if} \quad a_{22} \neq 0, \quad (7.3.18)$$

$$Y(x) = Y_1(x)\left[I - (1+m^2)P - \int_0^x Y_1^*(t)Y_1(t)\,dt\right]^{-1}, \quad (7.3.18')$$

$$\text{if} \quad a_{22} = 0.$$

This lemma is an immediate consequence of Theorem 6.2.1 and the relations (7.3.9′), (7.3.12′), and (7.3.13).

Lemma 7.3.4: *For any complex value of* $\lambda \neq 0\,(\operatorname{Im}\lambda \leq 0)$, *the equation* (7.1.0) *has a fundamental system of solutions* $E(x, \lambda)$ *and* $E^{(1)}(x, \lambda)$ *with the respective asymptotic behavior*

$$E(x, \lambda) = e^{-i\lambda x}[I + o(1)], \quad E^{(1)}(x, \lambda) = e^{i\lambda x}[I + o(1)] \qquad (x \to \infty).$$

Furthermore, these relations may be differentiated. The transformation (7.3.4) (*and its inverse* (7.3.17)) *preserves the asymptotic form of the solutions of* (7.1.0) (*or* (7.3.A)) *at infinity.*

Proof: Since the equation (7.3.A) has no singularity at infinity, according to Theorem 1.4.1 there exists a fundamental system of solutions $E_1(x, \lambda)$ and $E_1^{(1)}(x, \lambda)$ of this equation for any $\lambda \neq 0$ $(\operatorname{Im}\lambda \leq 0)$ such that

$$E_1(x, \lambda) = e^{-i\lambda x}[I + o(1)], \quad E_1^{(1)}(x, \lambda) = e^{i\lambda x}[I + o(1)] \qquad (x \to \infty).$$

Moreover, these relations may be differentiated. Applying (7.3.17) to these solutions, we obtain solutions $E(x, \lambda)$ and $E^{(1)}(x, \lambda)$ of equation (7.1.0), where by (7.3.2) and (7.3.6)

$$\begin{aligned} E(x, \lambda) &= [I + O(x^{-1})]E_1(x, \lambda) && (x \to \infty), \\ E^{(1)}(x, \lambda) &= [I + O(x^{-1})]E^{(1)}(x, \lambda) && (x \to \infty). \end{aligned}$$

From this, all of the assertions of the lemma follow.

2. We now consider in more detail the case where $a_{22} \neq 0$, i.e., $Y(x) \to \infty$ when $x \to 0$. According to Lemma 7.3.1, the equation (7.3.A) then has no singularities:

$$\int_0^\infty x^{1+\theta}\,|V_1(x)|\,dx < \infty \qquad (|\theta| < \varepsilon).$$

Therefore as proved in the first part of the book, the boundary-value problem (7.3.A)—(0) has a finite number of negative eigenvalues λ_k^2 $(k = 1, 2, \ldots, p)$ and a continuous spectrum over the entire positive axis $(\lambda^2 > 0)$.

The normalized eigenfunctions $U_1(x, \lambda)$ of this problem, for which Parseval's equality holds:

$$\int_0^\infty \Phi(x)\Psi^*(x)dx = \sum_{k=1}^{p} U_1(\Phi;\ \lambda_k)U_1^*(\Psi;\ \lambda_k) + \\ + \frac{1}{2\pi}\int_0^\infty U_1(\Phi;\ \lambda)U_1^*(\Psi;\ \lambda)\,d\lambda,$$

may be so chosen that as $x \to \infty$

$$U_1(x, \lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0),$$
$$U_1(x, \lambda_k) = e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 1, 2, \ldots, p).$$

Here $S(\lambda)$ is the scattering matrix and $M_1, M_2, \ldots, M_p$ the normalization matrices of the problem in question.

Now applying the inverse transformation (7.3.17) to $U_1(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1, 2, \ldots, p$), we obtain solutions $U(x, \lambda)$ of (7.1.0). According to Lemma 7.3.4, $U(x, \lambda)$ has the same asymptotic behavior at infinity as $U_1(x, \lambda)$. Making use of (7.3.10) and the relations $U_1(x, \lambda) = O(x)$ and $U_1'(x, \lambda) = O(1)$ when $x \to 0$ (see Lemma 7.3.2), we easily find that

$$\lim_{x \to 0} W\{Y_1^*(x), U_1(x, \lambda)\} = 0.$$

And therefore the inverse transformation (7.3.17) as applied to $U_1(x, \lambda)$ is expressible in the form (see Lemma 6.3.1)

$$U(x, \lambda) = U_1(x, \lambda) + Y(x) \int_0^x Y_1^*(t) U_1(t, \lambda)\, dt. \qquad (7.3.19)$$

Hence, we immediately see that $U(0, \lambda) = U_1(0, \lambda) = 0$, and by Theorem 6.3.1 and equation (7.3.18) we then have

$$\int_0^\infty \Phi(x)\Phi^*(x)\, dx = \sum_{k=1}^{p} U(\Phi; \lambda_k) U^*(\Phi; \lambda_k) +$$
$$+ \frac{1}{2\pi} \int_0^\infty U(\Phi; \lambda) U^*(\Phi; \lambda)\, d\lambda - Y(\Phi)(I - P)Y^*(\Phi).$$

Here, $\Phi(x)$ is any matrix with elements in $L^2(0, \infty)$ that vanish in the neighborhood of zero and infinity, and

$$U(\Phi; \lambda) = \int_0^\infty \Phi(x) U(x, \lambda)\, dx, \qquad Y(\Phi) = \int_0^\infty \Phi(x) Y(x)\, dx.$$

Since $Y(x)P = Y(x)$, we have $Y(\Phi)(I - P) = 0$, and hence for all $\Phi(x)$ of the stated form, the following Parseval equality holds:

$$\int_0^\infty \Phi(x)\Phi^*(x)\, dx = \sum_{k=1}^{p} U(\Phi; \lambda_k) U^*(\Phi; \lambda_k) +$$
$$+ \frac{1}{2\pi} \int_0^\infty U(\Phi; \lambda) U^*(\Phi; \lambda)\, d\lambda. \qquad (7.3.20)$$

As usual, one can now show that (7.3.20) is valid for any matrix $\Phi(x)$ with elements in $L^2(0, \infty)$.

Thus, we have proved

THEOREM 7.3.1: *If* $a_{22} \neq 0$, *then the boundary-value problem* (7.1.0)—(0) *has a finite number of negative eigenvalues* ($\lambda_k^2 < 0$; $k = 1, 2, \ldots, p$) *and a continuous spectrum over the entire positive axis* ($\lambda^2 > 0$). *There exists a complete set of normalized matrix eigenfunctions* $U(x, \lambda)$ *of this problem satisfying Parseval's equality* (7.3.20). *The eigenfunctions possess the following asymptotic behavior at infinity:*

$$U(x, \lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0),$$

$$U(x, \lambda_k) = e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 1, 2, \ldots, p),$$

where $S(\lambda) = S^*(-\lambda)$ *is a unitary matrix (the scattering matrix), and the* M_k *are non-negative Hermitian matrices (the normalization matrices).*

The aggregate of quantities $S(\lambda)$, λ_k^2, and $M_k (k = 1, 2, \ldots, p)$ will henceforth be called the *scattering data* of problem (7.1.0)—(0).

COROLLARY. If $a_{22} \neq 0$, then the scattering data of the problem (7.1.0)—(0) is simultaneously the scattering data of the non-singular problem (7.3.A)—(0) and therefore has the properties I_s, II_s, III_s, IV, and V (see §1, Chapter V). In this connection, the matrix $F_s(t)$, of which $I - S(\lambda)$ is the Fourier transform, is summable over the entire real axis (see Theorem 5.6.2) and satisfies a stronger condition than II_s, namely,

$$\int_0^\infty t^{1+\theta} \,|\, F_s'(t) \,|\, dt < \infty \qquad (|\,\theta\,| < \varepsilon).$$

This result follows from the analogous condition for $V_1(x)$ and the inequality (3.2.7).

REMARK. The scattering data for the case under consideration has one further property which is implied by the matrix $Y_1(x)$. In fact, this matrix is a solution of (7.3.A) when $\lambda = 0$, and thus according to (7.3.10), $Y_1(x) = -3\bar{a}_{22}^{-1}G_1(x)P$, where $G_1(x)$ is the solution of (7.3.A) with $\lambda = 0$ that satisfies the initial conditions $G_1(0) = 0$ and $G_1'(0) = I$. Therefore, as $x \to \infty$ (see Lemma 1.5.1)

$$Y_1(x) = -3\bar{a}_{22}^{-1}xE_1^*(0)P + o(x^{1-\theta})P \qquad (\theta < \varepsilon),$$

and from this and (7.3.6) it follows that $-3\bar{a}_{22}^{-1}E_1^*(0)P = -3P$; in other words $PE_1(0) = a_{22}P$. Using the special solution (1.3.2), we finally obtain

$$PE_1(0) = PE_1(x, \lambda)\Big|_{\substack{x=0\\ \lambda=0}} = P\left[I + \int_0^\infty K_1(0, t)\,dt\right] = a_{22}P \neq 0. \tag{7.3.21}$$

Since $K_1(0, t)$ is determined by the fundamental equation (3.1.12) based on the scattering data of the problem (7.3.A)—(0), and since this data coincides with the scattering data of the problem (7.1.0)—(0), the relation (7.3.21) is an additional condition on the scattering data of the problem (7.1.0)—(0) for the case $a_{22} \neq 0$.

§ 4. The Second Transformation

1. If $a_{22} = 0$ (i.e., $Y(x)$ is bounded at $x = 0$), then after the first transformation (7.3.4) the resulting equation (7.3.A) has exactly the same singularity at $x = 0$ as does the given equation (7.1.0). The elimination of this singularity will be the subject of this section.

Since (7.3.A) is non-singular at infinity, by Theorem 1.4.1, a solution $E_1(x, \lambda)$ of this equation exists for any λ in the lower half-plane with the following asymptotic behavior at infinity:

$$E_1(x, \lambda) = e^{-i\lambda x}[I + o(1)],\ E_1'(x, \lambda) = -i\lambda e^{-i\lambda x}[I + o(1)]. \tag{7.4.1}$$

On the other hand, by the remark of Theorem 7.2.2, this equation has a fundamental system of solutions $\Psi_1^{(1)}(x, \lambda)$ and $\Psi_1^{(2)}(x, \lambda)$ such that for any $\theta < \varepsilon$

$$\begin{aligned} \Psi_1^{(1)}(x, \lambda) &= \begin{pmatrix} 1 + o(x^\theta) & o(x^{-2+\theta}) \\ o(x^\theta) & x^{-2} + o(x^{-2+\theta}) \end{pmatrix} \quad (x \to 0), \\ \Psi_1^{(2)}(x, \lambda) &= \begin{pmatrix} x + o(x^{1+\theta}) & o(x^{3+\theta}) \\ o(x^{1+\theta}) & x^3 + o(x^{3+\theta}) \end{pmatrix} \quad (x \to 0). \end{aligned} \tag{7.4.2}$$

Moreover, these relations may be differentiated.

Therefore,

$$E_1(x, \lambda) = \Psi_1^{(1)}(x, \lambda)C_1(\lambda) + \Psi_1^{(2)}(x, \lambda)C_2(\lambda),$$

where $C_1(\lambda)$ and $C_2(\lambda)$ are matrices not depending on x.

If λ is some pure maginary in the lower half-plane such that $\det C_1(\lambda) = 0$, then the last equation shows that λ^2 is an eigenvalue of the problem (7.3.A)—(0).

Let us now choose $\lambda_0 = -ia_0$ ($a_0 > 0$) so that λ_0^2 is distinct from all of the eigenvalues of both of the problems (7.3.A)—(0) and (7.1.0)—(0).[1] Thus $\det C_1(\lambda_0) \neq 0$ and

$$Y_1(x, \lambda_0) = E_1(x, \lambda_0)C_1^{-1}(\lambda_0)P = \Psi_1^{(1)}(x, \lambda_0)P + \\ + \Psi_1^{(2)}(x, \lambda_0)C_2(\lambda_0)C_1^{-1}(\lambda_0)P$$

is a solution of equation (7.3.A) for $\lambda = \lambda_0$ having the asymptotic behavior

$$Y_1(x, \lambda_0) = e^{-a_0 x}O(1)P \qquad (x \to \infty), \tag{7.4.3}$$

$$Y_1(x, \lambda_0) = [x^{-2}I + o(x^{-2+\theta})]P \qquad (x \to 0;\ \theta < \varepsilon) \tag{7.4.4}$$

(see (7.4.1) and (7.4.2)); moreover, these relations may be differentiated.

Now define

$$Y_2(x, \lambda_0) = Y_1(x, \lambda_0)\left[I + \int_x^\infty Y_1^*(t, \lambda_0)Y_1(t, \lambda_0)\,dt\right]^{-1} \tag{7.4.5}$$

and consider the transformation

$$\Phi_2(x, \lambda) = \Phi_1(x, \lambda) + \\ + Y_2(x, \lambda_0)W\{Y_1^*(x, \lambda_0), \Phi_1(x, \lambda)\}(\lambda_0^2 - \lambda^2)^{-1}. \tag{7.4.6}$$

By Theorem 6.2.1, it transforms any solution $\Phi_1(x, \lambda)$ ($\lambda \neq \lambda_0$) of equation (7.3.A) into a solution $\Phi_2(x, \lambda)$ of

$$Y'' - V_2(x)Y + \lambda^2 Y = 0, \qquad 0 < x < \infty, \tag{7.4.B}$$

where

$$V_2(x) = V_1(x) + 2[Y_2(x, \lambda_0)Y_1^*(x, \lambda_0)]'. \tag{7.4.7}$$

A simple computation using (7.4.3) and (7.4.5) shows that

$$[Y_2(x, \lambda_0)Y_1^*(x, \lambda_0)]' = e^{-2a_0 x}O(1) \qquad (x \to \infty),$$

and hence it is apparent that for any $\theta < \varepsilon$, $V_2(x)$ satisfies the condition

$$\int^\infty x^{1+\theta}\,|V_2(x)|\,dx < \infty.$$

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[1] It is evidently always possible to do this since each of the indicated boundary-value problems has at most a finite number of eigenvalues.

By (7.4.4), we further have

$$\begin{aligned}\left[I+\int_x^\infty Y_1^*(t,\ \lambda_0)Y_1(t,\ \lambda_0)\,dt\right]^{-1} &= \\ &= I-P+3x^3P+Po(x^{3+\theta})P \qquad (x\to 0),\end{aligned} \tag{7.4.8}$$

$$Y_2(x,\ \lambda_0) = [3xI+o(x^{1+\theta})]P \qquad (x\to 0), \tag{7.4.9}$$

and these relations may be differentiated.

Therefore,

$$[Y_2(x,\ \lambda_0)Y_1^*(x,\ \lambda_0)]' = -3x^{-2}P+o(x^{-2+\theta}) \qquad (x\to 0)$$

and according to (7.4.7),

$$V_2(x) = V_1(x)-6x^{-2}P+o(x^{-2+\theta}) \qquad (x\to 0).$$

Thus, for any $\theta < \varepsilon$

$$\int_0 x^{1-\theta}\,|\,V_2(x)\,|\,dx<\infty,$$

and we have therefore proved

LEMMA 7.4.1: *Any matrix $\Phi_2(x,\lambda)$ obtained from a solution $\Phi_1(x,\lambda)$ of* (7.3.A) ($\lambda \neq \lambda_0$) *by means of the transformation* (7.4.6) *is a solution of*

$$Y''-V_2(x)Y+\lambda^2Y = 0, \qquad 0<x<\infty, \tag{7.4.B}$$

where for any $|\,\theta\,| < \varepsilon$

$$\int_0^\infty x^{1+\theta}\,|\,V_2(x)\,|\,dx<\infty.$$

REMARK. By using (7.4.3) and (7.4.5), we can easily show that the transformation (7.4.6) and its inverse preserve the asymptotic form of the solutions at infinity. Furthermore, if the solution $\Phi_1(x,\lambda)$ of equation (7.3.A) behaves like $o(e^{|\lambda_0|x})$ at infinity, then $\lim\limits_{x\to\infty} W\{Y_1^*(x,\lambda_0),\ \Phi_1(x,\lambda)\} = 0$, and as a consequence of Lemma 6.3.1, the transformation (7.4.6) is expressible as

$$\Phi_2(x,\ \lambda) = \Phi_1(x,\ \lambda)-Y_2(x,\ \lambda_0)\int_x^\infty Y_1^*(t,\lambda_0)\Phi_1(t,\ \lambda)\,dt. \tag{7.4.10}$$

LEMMA 7.4.2: *Let $\Phi_2^{(0)}(x,\lambda)$ and $Y_2(x)$ be the matrices obtained by applying the transformation* (7.4.6) *to $\Phi_1^{(0)}(x,\lambda)$* (see Lemma 7.3.2) *and $Y_1(x)$, respectively.*

Then as $x \to 0$

$$\Phi_2^{(0)}(x, \lambda) = O(x), \quad \textit{if} \quad a_{22} = a_{12} = 0,$$

$$\Phi_2^{(0)}(x, \lambda) = -|a_{12}|^2 m^2 \lambda^{-2}(I-P) + o(x^{\theta}), \quad \textit{if} \quad a_{22} = 0,\ a_{12} \neq 0,$$

and for any values of λ *and* μ *distinct from zero and* $\pm\lambda_0$,

$$\lim_{x\to\infty} W\{\Phi_2^{(0)*}(x, \lambda),\ \Phi_2^{(0)}(x, \mu)\} = |a_{12}|^2 m^2 (\mu^{-2} - \bar{\lambda}^{-2})(I-P),$$

$$\lim_{x\to\infty} W\{\Phi_2^{(0)*}(x, \lambda),\ Y_2(x)\} = -Y_2(0) = -Y_1(0).$$

Proof: According to Lemma 6.3.1, the transformation (7.4.6) can be written in the form

$$\Phi_2^{(0)}(x, \lambda) = \Phi_1^{(0)}(x, \lambda) -$$
$$-Y_2(x, \lambda_0)\left[\int_x^1 Y_1^*(t, \lambda_0)\Phi_1^{(0)}(t, \lambda)\,dt + A_1(\lambda)\right],$$

in which $A_1(\lambda)$ is a matrix not depending on x. Hence by using (7.4.4), (7.4.9), and Lemma 7.3.2, we find that $\Phi_2^{(0)}(x, \lambda) = O(x)$ at $x = 0$ for $a_{22} = a_{12} = 0$. As a consequence,

$$\lim_{x\to 0} W\{\Phi_2^{(0)*}(x, \lambda),\ \Phi_2^{(0)}(x, \mu)\} = 0.$$

However, if $a_{22} = 0$ but $a_{12} \neq 0$, then by the same Lemma 7.3.2, we have

$$\Phi_2^{(0)}(x, \lambda) = \Phi^{(0)}(x, \lambda) + O(x^2) -$$
$$-Y_2(x, \lambda_0)\left\{\int_x^1 Y_1^*(t, \lambda_0)\,[\Phi^{(0)}(t, \lambda) + O(t^2)]\,dt + A_1(\lambda)\right\} +$$
$$+\left[Y_1(x) - Y_2(x, \lambda_0)\int_x^1 Y_1^*(t, \lambda_0)Y_1(t)\,dt\right]B_0(\lambda),$$

and hence by virtue of (7.4.4), (7.4.9) and (7.3.1),

$$\Phi_2^{(0)}(x, \lambda) = x[I-P] + \left[Y_1(x) - Y_2(x, \lambda_0)\int_x^\infty Y_1^*(t, \lambda_0)Y_1(t)\,dt\right]B_0(\lambda)$$
$$+Y_2(x, \lambda_0)A_2(\lambda) + o(x^{1+\theta}) \qquad (x \to 0).$$

Or

$$\Phi_2^{(0)}(x, \lambda) = x[I-P] + Y_2(x)B_0(\lambda) +$$
$$+Y_2(x, \lambda_0)A_2(\lambda) + o(x^{1+\theta}) \qquad (x \to 0), \tag{7.4.11}$$

where

$$Y_2(x) = Y_1(x) - Y_2(x, \lambda_0) \int_x^\infty Y_1^*(t, \lambda_0) Y_1(t)\, dt, \tag{7.4.12}$$

$$A_2(\lambda) = -\int_0^1 Y_1^*(t, \lambda_0)\, [\Phi^{(0)}(t, \lambda) + O(t^2)]\, dt +$$

$$+ \int_1^\infty Y_1^*(t, \lambda_0) Y_1(t)\, dt \cdot B_0(\lambda) - A_1(\lambda).$$

A comparison of (7.4.12) and (7.4.10) shows that $Y_2(x)$ is the image of $Y_1(x)$ under the transformation (7.4.6) and is therefore a solution of (7.4.B) for $\lambda = 0$.

The application of (7.3.14), (7.4.4), (7.4.9), and of the definition of $B_0(\lambda)$ (see Lemma 7.3.2) to (7.4.12) then yields

$$Y_2(x) B_0(\lambda) = -|a_{12}|^2 m^2 \lambda^{-2} (I - P) + o(x^\theta) \qquad (\theta < \varepsilon), \tag{7.4.13}$$

as $x \to 0$, and this relation may be differentiated.

The equations (7.4.11), (7.4.9) and (7.4.13) now show that

$$\Phi_2^{(0)}(x, \lambda) = -|a_{12}|^2 m^2 \lambda^{-2} (I - P) + o(x^\theta) \qquad (x \to 0,\ \theta > \varepsilon),$$

and also that

$$\lim_{x \to 0} W\{\Phi_2^{(0)*}(x, \lambda),\ \Phi_2^{(0)}(x, \mu)\} = |a_{12}|^2 m^2 (\mu^{-2} - \overline{\lambda}^{-2}) (I - P) +$$

$$+ \lim_{x \to 0} \big[W\{B_0^*(\lambda) Y_2^*(x),\ Y_2(x, \lambda_0) A_2(\mu)\} +$$

$$+ W\{A_2^*(\lambda) Y_2^*(x, \lambda_0),\ Y_2(x) B_0(\mu)\} + W\{B_0(\lambda) Y_2^*(x),\ Y_2(x) B_0(\mu)\}\big].$$

But by (7.4.9) and (7.4.13),

$$\lim_{x \to 0} W\{Y_2^*(x),\ Y_2(x, \lambda_0)\} = \lim_{x \to 0} W\{Y_2^*(x, \lambda_0),\ Y_2(x)\} = 0,$$

and since the Wronskian $W\{Y_2^*(x), Y_2(x)\}$ is independent of x, there results

$$\lim_{x \to 0} W\{\Phi_2^{(0)*}(x, \lambda),\ \Phi_2^{(0)}(x, \mu)\} = |a_{12}|^2 m^2 (\mu^{-2} - \overline{\lambda}^{-2}) (I - P) +$$

$$+ B_0^*(\lambda) W\{Y_2^*(x),\ Y_2(x)\} B_0(\mu).$$

We now note that

$$Y_2(x) = Y_1(x) + O(x e^{-2 a_0 x}) \qquad (x \to \infty)$$

by virtue of (7.4.12), and therefore

$$W\{Y_2^*(x),\ Y_2(x)\} = \lim_{x\to\infty} W\{Y_2^*(x),\ Y_2(x)\} \equiv W\{Y_1^*(x),\ Y_1(x)\}.$$

However, by Lemma 6.2.2, $W\{Y_1^*(x),\ Y_1(x)\} \equiv 0$, and thus

$$\lim_{x\to 0} W\{\Phi_2^{(0)*}(x,\lambda),\ \Phi_2^{(0)}(x,\mu)\} = |a_{12}|^2 m^2(\mu^{-2} - \bar{\lambda}^{-2})(I-P).$$

The result

$$\lim_{x\to 0} W\{\Phi_2^{(0)*}(x,\ \lambda),\ Y_2(x)\} = -Y_2(0) = -Y_1(0)$$

can be derived from (7.4.11) and (7.4.12) in a completely analogous way.

Lemma 7.4.3: *The transformation inverse to* (7.4.6) *is given by*

$$\begin{gathered}\Phi_1(x,\ \lambda) = \Phi_2(x,\ \lambda) - \\ -Y_1(x,\ \lambda_0)W\{Y_2^*(x,\ \lambda_0),\ \Phi_2(x,\ \lambda)\}(\lambda_0^2-\lambda^2)^{-1},\end{gathered} \tag{7.4.14}$$

where

$$Y_1(x,\ \lambda_0) = Y_2(x,\ \lambda_0)\left[I-P+\int_0^x Y_2^*(t,\ \lambda_0)Y_2(t,\ \lambda_0)\,dt\right]^{-1}, \tag{7.4.15}$$

$$\int_0^\infty Y_2^*(t,\ \lambda_0)Y_2(t,\ \lambda_0)\,dt = P. \tag{7.4.16}$$

Proof: The expressions (7.4.14) and (7.4.15) are immediate consequences of Theorem 6.2.1 and equations (7.4.5) and (7.4.8). In this connection, we also have

$$\begin{gathered}\left[I+\int_x^\infty Y_1^*(t,\ \lambda_0)Y_1(t,\ \lambda_0)\,dt\right]^{-1} = \\ = I-P+\int_0^x Y_2^*(t,\ \lambda_0)Y_2(t,\ \lambda_0)\,dt.\end{gathered} \tag{7.4.17}$$

Letting x tend to infinity, we find that

$$I = I-P+\int_0^\infty Y_2^*(t,\ \lambda_0)Y_2(t,\ \lambda)\,dt,$$

and this result is equivalent to (7.4.16).

2. Let us now consider the boundary-value problem (7.4.B)—(0) defined by the equation (7.4.B) and the boundary condition $Y(0,\ \lambda) = 0$. By Lemma 6.2.2, the matrix $Y_2(x,\ \lambda_0)$ is a solution of (7.4.B) for $\lambda = \lambda_0$. Furthermore, it vanishes at $x = 0$ (see (7.4.9)) and its elements belong to $L^2(0, \infty)$ (see (7.4.3) and (7.4.5)). As a consequence, λ_0^2 is an eigenvalue of the boundary-value problem in question, and any non-zero column of $Y_2(x, \lambda_0)$ is a corresponding eigenfunction in normalized form by virtue of (7.4.16). Since (7.4.B) has no singularities, the problem (7.4.B)—(0) has possibly a finite number of additional negative eigenvalues ($\lambda_k^2 < 0$; $k = 1, 2, \ldots, p-1$) and a continuous spectrum over the entire positive axis ($\lambda^2 > 0$).

Let the matrices $U_2(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 0, 1, 2, \ldots, p-1$) form a complete set of normalized eigenfunctions of this problem, for which the Parseval equality holds:

$$\int_0^\infty \Phi(x)\Phi^*(x)\,dx = \sum_{k=0}^{p-1} U_2(\Phi;\ \lambda_k)U_2^*(\Phi;\ \lambda_k) + \frac{1}{2\pi}\int_0^\infty U_2(\Phi;\ \lambda)U_2^*(\Phi;\ \lambda)\,d\lambda, \tag{7.4.18}$$

and such that as $x \to \infty$

$$U_2(x, \lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0), \tag{7.4.19}$$

$$U_2(x, \lambda_k) = e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 0, 1, \ldots, p-1). \tag{7.4.20}$$

Here, $S(\lambda)$, λ_k^2, and M_k comprise the scattering data for the problem under consideration. From the above properties of $Y_2(x, \lambda_0)$ we have

$$U_2(x, \lambda_0) = Y_2(x, \lambda_0)U,$$

where U is a fixed unitary matrix.

To $U_2(x, \lambda)$, we now apply the transformation (7.4.14) which in the present case can be expressed as

$$U_1(x, \lambda) = U_2(x, \lambda) - Y_1(x, \lambda_0)\int_0^x Y_2^*(t, \lambda_0)U_2(t, \lambda)\,dt. \tag{7.4.21}$$

In fact, Lemma 6.3.1 is applicable since $U_2(x, \lambda)$ and $Y_2(x, \lambda_0)$ are both $O(x)$ as $x \to 0$, and thus $\lim\limits_{x\to 0} W\{Y_2^*(x, \lambda_0),\ U_2(x, \lambda)\} = 0$. By Lemma 7.4.3 and the remark of Lemma 6.3.1, the matrices $U_1(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 0, 1, 2, \ldots, p-1$) are solutions of equation (7.3.A). Because of the remark of Lemma 7.4.1, they have the same asymptotic behavior at infinity as $U_2(x, \lambda)$ for $\lambda \neq \lambda_0$, i.e. they are asymptotically equal to the right-hand sides of (7.4.19) and (7.4.20).

Taking into consideration (7.4.4), (7.4.9), and the fact that $U_2(x, \lambda) = O(x)$ at $x = 0$, we find from (7.4.21) that $U_1(0, \lambda) = 0$.

Finally, (7.4.18), (7.4.21), and (7.4.15) in conjunction with Theorem 6.3.1 yield

$$\begin{aligned}\int_0^\infty \Phi(x)\Phi^*(x)\,dx &= \sum_{k=0}^{p-1} U_1(\Phi;\ \lambda_k)U_1^*(\Phi;\ \lambda_k)+\\ &+\frac{1}{2\pi}\int_0^\infty U_1(\Phi;\ \lambda)U_1^*(\Phi;\ \lambda)\,d\lambda+Y_1(\Phi;\ \lambda_0)(I-P)Y_1^*(\Phi;\ \lambda_0),\end{aligned} \tag{7.4.22}$$

where $\Phi(x)$ is any matrix with elements in $L^2(0, \infty)$ vanishing in the neighborhood of zero and infinity, and

$$U_1(\Phi;\ \lambda)= \int_0^\infty \Phi(x)U_1(x,\ \lambda)\,dx, \quad Y_1(\Phi;\ \lambda_0)= \int_0^\infty \Phi(x)Y_1(x, \lambda_0)\,dx.$$

We now note that $Y_1(x, \lambda_0) = Y_1(x, \lambda_0)\,P$ and hence,

$$Y_1(\Phi;\ \lambda_0)(I-P) = Y_1(\Phi;\ \lambda_0)P(I-P) = 0.$$

Furthermore,

$$\begin{aligned}U_1(x,\ \lambda_0) &= U_2(x,\ \lambda_0)-Y_1(x,\ \lambda_0)\int_0^x Y_2^*(t,\ \lambda_0)U_2(t,\ \lambda_0)\,dt =\\ &= \left[Y_2(x, \lambda_0)-Y_1(x, \lambda_0)\int_0^x Y_2^*(t,\ \lambda_0)Y_2(t,\ \lambda_0)\,dt\right]U =\\ &= Y_1(x,\ \lambda_0)\left\{\left[I+\int_x^\infty Y_1^*(t,\ \lambda_0)Y_1(t,\ \lambda_0)\,dt\right]^{-1}-\right.\\ &\qquad\left.-\int_0^x Y_2^*(t,\ \lambda_0)Y_2(t,\ \lambda_0)\,dt\right\}U,\end{aligned}$$

and according to (7.4.17) we conclude from this that

$$U_1(x,\ \lambda_0) = Y_1(x,\ \lambda_0)(I-P)U \equiv 0.$$

Hence, the formula (7.4.22) actually has the following form:

$$\begin{aligned}\int_0^\infty \Phi(x)\Phi^*(x)\,dx &= \sum_{k=1}^{p-1} U_1(\Phi;\ \lambda_k)U_1^*(\Phi;\ \lambda_k)+\\ &+\frac{1}{2\pi}\int_0^\infty U_1(\Phi;\ \lambda)U_1^*(\Phi;\ \lambda)\,d\lambda.\end{aligned} \tag{7.4.23}$$

Thus, the matrices $U_1(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 1, 2, \ldots, p-1)$ form a complete set of normalized eigenfunctions of the boundary-value problem (7.3.A)—(0), and the Parseval equality (7.4.23) holds (which by the usual means can be extended to any matrix $\Phi(x)$ with elements in $L^2(0, \infty)$). At infinity, these eigenfunctions have the asymptotic behavior described by (7.4.19) and (7.4.20).[2]

3. Let us now apply to $U_1(x, \lambda)$ the transformation (7.3.17) inverse to (7.3.4), restricting ourselves at this point to the case $a_{22} = a_{12} = 0$ (i.e., $Y(0) = 0$). According to (7.3.3) and (7.3.16), $Y_1(x) = O(x)$ as $x \to 0$ so that $\lim_{x\to 0} W\{Y_1^*(x),\ U_1(x, \lambda)\} = 0$. Therefore the transformation (7.3.17) may be applied to $U_1(x, \lambda)$ in the form (7.3.19).

As a result we obtain the matrices $U(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 1, 2, \ldots, p-1)$ which are solutions of (7.1.0) vanishing at $x = 0$ and having, by Lemma 7.3.4, the same asymptotic behavior at infinity as the matrices $U_1(x, \lambda)$. Moreover, since Parseval's equality (7.4.23) holds for the matrices $U_1(x, \lambda)$, by (7.3.19), (7.3.18′), and Theorem 6.3.1 we have

$$\int_0^\infty \Phi(x)\Phi^*(x)\,dx = \sum_{k=1}^{p-1} U(\Phi;\ \lambda_k)U^*(\Phi;\ \lambda_k) + $$
$$+\frac{1}{2\pi}\int_0^\infty U(\Phi;\ \lambda)U^*(\Phi;\ \lambda)\,d\lambda - Y(\Phi)\,[I - P - m^2P]Y^*(\Phi).$$

But $Y(x)P = Y(x)$, and hence $Y(\Phi)(I-P) = 0$. Therefore setting $U(x, 0) = Y(x)\,m$, we obtain

$$\int_0^\infty \Phi(x)\Phi^*(x)\,dx = \sum_{k=1}^{p} U(\Phi;\ \lambda_k)U^*(\Phi;\ \lambda_k) + $$
$$+\frac{1}{2\pi}\int_0^\infty U(\Phi;\ \lambda)U^*(\Phi;\ \lambda)\,d\lambda, \tag{7.4.24}$$

with $\lambda_p = 0$.

Thus we have proved

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[2] We have here taken into consideration the fact that $\lambda_k \neq \lambda_0 (k = 1,2, \ldots, p-1)$ since λ_0^2 was chosen to be distinct from all of the eigenvalues of the problem (7. 3. A)—(0).

THEOREM 7.4.1: *If* $a_{22} = a_{12} = 0$ *(i.e.,* $Y(0) = 0$*), then the boundary-value problem* (7.1.0)—(0) *has a finite number of negative eigenvalues* ($\lambda_k^2 < 0$, $k = 1, 2, \ldots, p-1$), *the eigenvalue* $\lambda_p^2 = 0$, *and a continuous spectrum over the entire positive axis. A complete set of normalized eigenfunctions* $U(x, \lambda)$ *exist for this problem having the asymptotic behavior*

$$\begin{aligned} U(x, \lambda) &= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0); \\ U(x, \lambda_k) &= e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 1, 2, \ldots, p-1); \qquad (x \to \infty), \\ U(x, \lambda_p) &= U(x, 0) = x^{-2}[M_p + o(1)] \qquad (M_p = mP), \end{aligned}$$

for which Parseval's equality (7.4.24) *holds. Here,* $S(\lambda) = S^*(-\lambda)$ *is a unitary matrix (the scattering matrix), and* M_k *are non-negative Hermitian matrices (the normalization matrices).*

COROLLARY. Suppose $a_{22} = a_{12} = 0$. Then we can obtain the scattering data of the non-singular problem (7.4.B)—(0) from that of the problem (7.1.0)—(0) by replacing $\lambda_p = 0$ and $M_p = mP$ by λ_0 and M_0, respectively. In this connection, the rank $M_p =$ rank $M_0 = 1$.

Hence, it follows, in particular, that the scattering matrix $S(\lambda)$ of the problem (7.1.0)—(0) satisfies the conditions I_s, II_s, and III_s for the case under consideration. Furthermore, the matrix $F_s(t)$, of which $I - S(\lambda)$ is the Fourier transform, is summable over the entire real axis (see Theorem 5.6.2) and satisfies a stronger condition than II_s, namely,

$$\int_0^\infty t^{1+\theta} \, | F_s'(t) | \, dt < \infty \qquad (|\theta| < \varepsilon),$$

since the analogous condition holds for $V_2(x)$ (see the inequality (3.2.7)).

In addition, there holds the following

LEMMA 7.4.4: *Let* $a_{22} = a_{12} = 0$, *and let* $S(\lambda)$, λ_k^2, M_k ($k = 1, 2, \ldots, p$; $\lambda_p^2 = 0$, $M_p = mP$) *be the scattering data for the boundary-value probleim* (7.1.0)—(0). *Let*

$$F_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [I - S(\lambda)] e^{i\lambda t} \, d\lambda,$$

$$F(t) = \sum_{k=1}^{p-1} M_k^2 e^{-|\lambda_k| t} + F_s(t).$$

Then the number of linearly independent vector solutions of

$$z(t)+\int_0^\infty z(\xi)F_s(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty \tag{7.4.25}$$

is equal to the sum of the ranks of the matrices $M_1, \ldots, M_p$, *and the equation*

$$x(t)+\int_0^\infty x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty \tag{7.4.26}$$

has a single non-trivial solution whose Fourier transform $\widetilde{x}(\lambda)$ *is such that*

$$\lim_{\lambda\to\infty} \lambda^2\widetilde{x}(\lambda)(I-P) = 0.$$

Proof: Since the quantities $S(\lambda)$, λ_k^2, M_k, λ_0^2, and M_0 constitute the scattering data for the non-singular problem (7.4.B)—(0), property *V* applies and the number of linearly independent solutions of (7.4.25) is equal to the sum of the ranks of the matrices M_0, $M_1, \ldots, M_{p-1}$. But rank M_0 = rank M_p and therefore the number of such solutions is also equal to the sum of the ranks of the matrices $M_1, M_2, \ldots, M_p$.

Reasoning now as we did in the proof of Theorem 3.5.1, we find that the Fourier transform $\widetilde{x}(\lambda)$ of any solution of (7.4.26) is given by

$$\widetilde{x}(\lambda) = \frac{2\lambda_0}{\lambda^2-\lambda_0^2}\widetilde{x}(\lambda_0)N_0E_2(\lambda) = \frac{1}{\lambda^2-\lambda_0^2}cN_0E_2(\lambda), \tag{7.4.27}$$

where $E_2(x, \lambda)$ is the special solution of (7.4.B), $E_2(\lambda) = E_2(0, \lambda)$, N_0 is the residue of $E_2^{-1}(\lambda)$ at its pole λ_0, and c is any fixed vector. Since N_0 and M_0 have the same rank of 1 and $E_2(\lambda)$ is non-singular for real values of $\lambda \neq 0$, formula (7.4.27) shows that equation (7.4.26) has a non-trivial solution which is unique to within a numerical factor.

To complete the proof of the lemma, we consider the normalized eigenfunction $Y_2(x, \lambda_0)$ of the problem (7.4.B)—(0).

From (7.4.9), we conclude that

$$Y_2(x, \lambda_0) = 3G_2(x, \lambda_0)P,$$

where $G_2(x, \lambda_0)$ is the solution of (7.4.B) satisfying the conditions $G_2(0, \lambda_0) = 0$ and $G_2'(0, \lambda_0) = I$. But according to the formula (3.1.10) for any normalized matrix eigenfunction of the non-singular boundary-value problem, we have

$$Y_2(x, \lambda_0) = 2\lambda_0 G_2(x, \lambda_0) N_0^* C_0,$$

where C_0 is some non-singular matrix. Hence, it follows that $3P = 2\lambda_0 N_0^* C_0$, and therefore, $N_0 = \frac{3}{2}\bar{\lambda}_0^{-1}(C_0^*)^{-1} P$ and $N_0 P = N_0$.

Making use of this result and the fact that $E_2(\lambda) \to I$ as $\lambda \to \infty$, we conclude from (7.4.27) that

$$\lim_{\lambda \to \infty} \lambda^2 \widetilde{x}(\lambda)(I-P) = 0.$$

The proof of the lemma is complete.

§ 5. The Third Transformation

1. In order to complete our investigation of the problem (7.1.0)—(0), it is still necessary to analyze the case where $a_{22} = 0$ but $a_{12} \neq 0$. In this instance, the successive application of the two transformations of the preceding sections converts solutions $\Phi^{(0)}(x, \lambda)$ of equation (7.1.0) into solutions $\Phi_2^{(0)}(x, \lambda)$ of the non-singular equation (7.4.B), where by Lemma 7.4.2,

$$\Phi_2^{(0)}(0, \lambda) = -|a_{12}|^2 m^2 \lambda^{-2}(I-P).$$

We shall now formulate one further transformation which will take any matrix $\Phi_2^{(0)}(x, \lambda)$ into a solution $\Phi_3^{(0)}(x, \lambda)$ of a new non-singular equation having the property $\Phi_3^{(0)}(0, \lambda) = 0$.

To this end, consider some arbitrary number α with $0 < \alpha < |\lambda_0|$ and such that $-\alpha^2$ is not an eigenvalue of the problem (7.1.0)—(0), and define

$$Y_2(x, -i\alpha) = \Phi_2^{(0)}(x, -i\alpha)(I-P), \tag{7.5.1}$$

$$\begin{aligned} &Y_3(x, -i\alpha) = \\ &= Y_2(x, -i\alpha)\left[\xi^2 I + \int_0^x Y_2^*(t, -i\alpha) Y_2(t, -i\alpha)\, dt\right]^{-1}. \end{aligned} \tag{7.5.2}$$

Here, ξ^2 is a positive quantity to be determined below. According to Lemma 7.4.2,

$$\begin{aligned} &W\{Y_2^*(x, -i\alpha), \quad Y_2(x, -i\alpha)\} = \\ &= (I-P) W\{\Phi_2^{(0)*}(x, -i\alpha), \ \Phi_2^{(0)}(x, -i\alpha)\}(I-P) = 0 \end{aligned}$$

and therefore, the transformation

$$\Phi_3(x,\ \lambda) = \Phi_2(x,\ \lambda) + \\ + Y_3(x,\ -i\alpha)W\{Y_2^*(x,\ -i\alpha),\ \Phi_2(x,\ \lambda)\}(\alpha^2+\lambda^2)^{-1} \qquad (7.5.3)$$

takes any solution $\Phi_2(x, \lambda)$ $(\lambda \neq -i\alpha)$ of (7.4.B) into a solution $\Phi_3(x, \lambda)$ of

$$Y'' - V_3(x)Y + \lambda^2 Y = 0 \qquad 0 < x < \infty, \qquad (7.5.C)$$

in which

$$V_3(x) = V_2(x) - 2[Y_3(x,\ -i\alpha)Y_2^*(x,\ -i\alpha)]'. \qquad (7.5.4)$$

From the definition (7.5.1), it follows that

$$Y_2(x,\ -i\alpha) = |a_{12}|^2 m^2 \alpha^{-2}(I-P) + o(x^\theta) \qquad (x \to 0), \qquad (7.5.5)$$

and this relation may be differentiated. From this and (7.5.2), we then have

$$[Y_3(x,\ -i\alpha)Y_2^*(x,\ -i\alpha)]' = o(x^{-1+\theta}) \qquad (x \to 0).$$

Therefore for any $\theta < \varepsilon$

$$\int_0 x^{1-\theta}\,|\,V_3(x)\,|\,dx < \infty,$$

since the analogous inequality holds for $V_2(x)$.

In order to determine the behavior of $V_3(x)$ at infinity, we make use of the fundamental system of solutions $E_2(x, \lambda)$ and $E_2^{(1)}(x, \lambda)$ of (7.4.B) whose existence was established in Theorem 1.4.1. We have

$$\Phi_2^{(0)}(x,\ -i\alpha) = E_2(x,\ -i\alpha)C_1 + E_2^{(1)}(x,\ -i\alpha)C_2,$$

$$Y_2(x,\ -i\alpha) = E_2(x,\ -i\alpha)C_1(I-P) + \\ + E_2^{(1)}(x,\ -i\alpha)C_2(I-P), \qquad (7.5.6)$$

where C_1 and C_2 are constant matrices, C_2 being non-singular, since in the contrary case $(-i\alpha)^2 = -\alpha^2$ would be an eigenvalue of the problem (7.1.0)—(0).[3]

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[3] If $\det C_2 = 0$, then a non-zero vector a exists such that $C_2 a = 0$. Hence, the vector function $\Phi_2^{(0)}(x, -i\alpha)a$ decreases exponentially when $x \to \infty$. If the transformations (7. 4. 14) and (7. 3. 17) are applied in succession to $\Phi_2^{(0)}(x, -i\alpha)a$ there results the vector function $\Phi^{(0)}(x, -i\alpha)a$ which vanishes at $x = 0$ and decreases exponentially when $x \to \infty$.

For brevity, let us denote by $\varepsilon(x)$ any matrix $A(x)$ which tends to zero as $x \to \infty$ and satisfies the condition

$$\int^{\infty} x^{1+\theta}\,|\,A(x)\,|\,dx < \infty$$

for any $\theta < \varepsilon$. The application of Theorem 1.4.1 to (7.5.6) then leads to

$$\begin{aligned} Y_2(x,\ -i\alpha) &= e^{\alpha x}\left[I - \frac{1}{2\alpha}\int_x^{\infty} V_2(t)\,dt + \varepsilon(x)\right] C_2(I-P), \\ Y_2'(x,\ -i\alpha) &= \alpha Y_2(x,\ -i\alpha) + e^{\alpha x}\varepsilon(x)\,(I-P). \end{aligned} \tag{7.5.7}$$

Therefore as $x \to \infty$

$$\begin{aligned} &\xi^2 I + \int_0^x Y_2^*(t,\ -i\alpha)Y_2(t,\ -i\alpha)\,dt = \\ &= \xi^2 I + (2\alpha)^{-1}e^{2\alpha x}(I-P)\,[C_2^*C_2 + o(1)]\,(I-P), \end{aligned}$$

and as a consequence, the matrix

$$Z(x) = \left[\xi^2 I + \int_0^x Y_2^*(t,\ -i\alpha)Y_2(t,\ -i\alpha)\,dt\right]^{-1} \tag{7.5.8}$$

is asymptotically equal to

$$\begin{aligned} Z(x) &= \xi^{-2}P + 2\alpha c^{-1}e^{-2\alpha x}(I-P) + \\ &+ (I-P)o(e^{-2\alpha x})\,(I-P) \qquad (x \to \infty), \end{aligned} \tag{7.5.9}$$

where c is a positive number defined by the equation

$$c(I-P) = (I-P)C_2^*C_2(I-P). \tag{7.5.10}$$

Hence, by (7.5.2), and (7.5.7),

$$Y_3(x,\ -i\alpha) = 2\alpha c^{-1}e^{-\alpha x}[C_2 + o(1)]\,(I-P) \qquad (x \to \infty) \tag{7.5.11}$$

and

$$\begin{aligned} &[Y_3(x,\ -i\alpha)Y_2^*(x,\ -i\alpha)]' = [Y_2(x,\ -i\alpha)Z(x)]'Y_2^*(x,\ -i\alpha) + \\ &+ Y_2(x,\ -i\alpha)Z(x)Y_2^{*\prime}(x,\ -i\alpha) = [Y_2'(x,\ -i\alpha)Z(x) - \\ &- Y_2(x,\ -i\alpha)Z(x)Y_2^*(x,\ -i\alpha)Y_2(x,\ -i\alpha)Z(x)]Y_2^*(x,\ -i\alpha) + \\ &+ Y_2(x,\ -i\alpha)Z(x)Y_2^{*\prime}(x,\ -i\alpha) = \\ &= 2\alpha Y_2(x,\ -i\alpha)Z(x)\left\{I - \frac{1}{2\alpha}Y_2^*(x,\ -i\alpha)Y_2(x,\ -i\alpha)Z(x)\right\}Y_2^*(x,\ -i\alpha) + \\ &+ \varepsilon(x), \end{aligned} \tag{7.5.12}$$

since by (7.5.8),

$$Z'(x) = -Z(x)Y_2^*(x, -i\alpha)Y_2(x, -i\alpha)Z(x).$$

Applying (7.5.7) and taking into consideration that

$$\left[\int_x^\infty V_2(t)\,dt\right]^2 = \varepsilon(x),$$

we then find that

$$Y_2^*(x, -i\alpha)Y_2(x, -i\alpha) =$$

$$= e^{2\alpha x}(I-P)\left\{C_2^*\left[I-\frac{1}{2\alpha}\int_x^\infty V_2(t)\,dt\right]^2 C_2+\varepsilon(x)\right\}(I-P) =$$

$$= e^{2\alpha x}(I-P)\left\{C_2^*\left[I-\frac{1}{\alpha}\int_x^\infty V_2(t)\,dt\right]C_2+\varepsilon(x)\right\}(I-P);$$

$$\int_0^x Y_2^*(t, -i\alpha)Y_2(t, -i\alpha)\,dt = \text{const}+\int_1^x Y_2^*(t, -i\alpha)Y_2(t, -i\alpha)\,dt =$$

$$= \text{const}+(I-P)\left\{C_2^*\int_1^x e^{2\alpha t}\left[I-\frac{1}{\alpha}\int_t^\infty V_2(s)\,ds\right]dt\,C_2+\right.$$

$$\left.+\int_1^x e^{2\alpha t}\varepsilon(t)\,dt\right\}(I-P) = \text{const}+$$

$$+(I-P)\left\{\frac{e^{2\alpha x}}{2\alpha}C_2^*\left[I-\frac{1}{\alpha}\int_x^\infty V_2(t)\,dt\right]C_2-\frac{1}{2\alpha^2}C_2^*\int_1^x e^{2\alpha t}V_2(t)\,dt\,C_2+\right.$$

$$\left.+\int_1^x e^{2\alpha t}\varepsilon(t)\,dt\right\}(I-P)$$

(in the last step, we have performed integration by parts).

A comparison of these relations leads to the conslusion

$$\frac{1}{2\alpha}Y_2^*(x, -i\alpha)Y_2(x, -i\alpha) = \int_0^x Y_2^*(t, -i\alpha)Y_2(t, -i\alpha)\,dt+\mathbf{A}(x),$$

where

$$\mathbf{A}(x) = \text{const} + (I-P)\left\{ e^{2\alpha x}\varepsilon(x) + \frac{1}{2\alpha^2} C_2^* \int_1^x e^{2\alpha t} V_2(t)\, dt\, C_2 - \right.$$

$$\left. - \int_1^x e^{2\alpha t}\varepsilon(t)\, dt \right\} (I-P).$$

A straightforward computation, which we shall omit, shows that[4]

$$e^{-2\alpha x}\,\mathbf{A}(x) = \varepsilon(x).$$

Therefore

$$I - \frac{1}{2\alpha} Y_2^*(x, -i\alpha) Y_2(x, -i\alpha) Z(x) =$$

$$= I - \left[\int_0^x Y_2^*(t, -i\alpha) Y_2(t, -i\alpha)\, dt + \mathbf{A}(x) \right] Z(x) =$$

$$= [\xi^2 I - \mathbf{A}(x)] Z(x) = [\xi^2 I - e^{2\alpha x}\varepsilon(x)] Z(x)$$

(see (7.5.8)). Substituting this result in (7.5.12) and then applying the asymptotic relations (7.5.7) and (7.5.9), we find that

$$[Y_3(x, -i\alpha) Y_2^*(x, -i\alpha)]' = \varepsilon(x).$$

Therefore, the matrix $V_3(x)$ defined by (7.5.4) satisfies the same condition as does $V_2(x)$, namely,

$$\int^\infty x^{1+\theta} \,|\, V_3(x) \,|\, dx < \infty \qquad (\theta < \varepsilon).$$

We now show that the positive number ξ^2 in the expression (7.5.2) may be so chosen that any matrix $\Phi_3^{(0)}(x, \lambda)$ obtained from $\Phi_2^{(0)}(x, \lambda)$ by means of the transformation (7.5.3) ($\lambda \neq \lambda_0$, $\lambda \neq -i\alpha$) vanishes at $x = 0$. By Lemma 7.4.2 and the definitions (7.5.1) and (7.5.2), we have

$$\Phi_2^{(0)}(0, \lambda) = -\,|\, a_{12} \,|^2 m^2 \lambda^{-2} (I-P),$$

$$Y_3(0, -i\alpha) = |\, a_{12} \,|^2 m^2 \alpha^{-2} \xi^{-2} (I-P),$$

$$W\{Y_2^*(x, -i\alpha),\ \Phi_2^{(0)}(x, \lambda)\}\,|_{x=0} =$$

$$= (I-P) W\{\Phi_2^{(0)*}(x, -i\alpha),\ \Phi_2^{(0)}(x, \lambda)\}\,|_{x=0} =$$

$$= |\, a_{12} \,|^2 m^2 (\lambda^{-2} + \alpha^{-2}) (I-P),$$

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[4] Analogous computations were carried out in the proof of Theorem 1.4.1.

and therefore

$$\begin{gathered}\Phi_3^{(0)}(0,\ \lambda) = \Phi_2^{(0)}(0,\ \lambda)+\\ +Y_3(0,\ -i\alpha)W\{Y_2^*(x,\ -i\alpha),\ \Phi_2^{(0)}(x,\ \lambda)\}\big|_{x=0}(\alpha^2+\lambda^2)^{-1} =\\ = (-|a_{12}|^2m^2\lambda^{-2}+|a_{12}|^4m^4\alpha^{-4}\xi^{-2}\lambda^{-2})(I-P).\end{gathered}$$

Hence, if we choose

$$\xi^2 = |a_{12}|^2m^2\alpha^{-4} \tag{7.5.13}$$

in (7.5.2), then

$$\Phi_3^{(0)}(0,\ \lambda) = 0.$$

Denoting by $Y_3(x)$ the solution of (7.5.C) for $\lambda = 0$ obtained by applying (7.5.3) to $Y_2(x)$ and using the portion of Lemma 7.4.2 concerning $Y_2(x)$, we can show in an analogous way that if ξ^2 is given by (7.5.13), then $Y_3(0) = 0$.

Thus, the required transformation has been constructed, and we have

LEMMA 7.5.1: *Any matrix $\Phi_3(x, \lambda)$ obtained from asolution $\Phi_2(x, \lambda)$ of equation* (7.4.B) $(\lambda \neq -i\alpha)$ *by means of the transformation* (7.5.3) *is a solution of*

$$Y'' - V_3(x)Y + \lambda^2 Y = 0, \qquad 0 < x < \infty, \tag{7.5.C}$$

in which $V_3(x)$ satisfies the condition

$$\int_0^\infty x^{1+\theta}\,|V_3(x)|\,dx < \infty$$

(or any $|\theta| < \varepsilon$. If $\xi^2 = |a_{12}|^2m^2\alpha^{-4}$, then the matrices $\Phi_3^{(0)}(x, \lambda)$ $(\lambda \neq \lambda_0,\ \lambda \neq -i\alpha)$ and $Y_3(x)$ obtained from $\Phi_2^{(0)}(x, \lambda)$ and $Y_2(x)$ respectively by the indicated transformation both vanish at $x = 0$.

An immediate consequence of Theorem 6.2.1, Lemma 6.3.1, and formula (7.5.2) is the following proposition concerning the form of the inverse transformation.

LEMMA 7.5.2. *The transformation inverse to* (7.5.3) *is given by*

$$\begin{gathered}\Phi_2(x,\ \lambda) = \Phi_3(x,\ \lambda)-\\ -Y_2(x,\ -i\alpha)W\{Y_3^*(x,\ -i\alpha),\ \Phi_3(x,\ \lambda)\}(\alpha^2+\lambda^2)^{-1},\end{gathered} \tag{7.5.14}$$

where

$$\begin{gathered}Y_2(x,\ -i\alpha) =\\ = Y_3(x,\ -i\alpha)\left[\xi^{-2}P+\int_x^\infty Y_3^*(t,\ -i\alpha)Y_3(t,\ -i\alpha)\,dt\right]^{-1}.\end{gathered} \tag{7.5.15}$$

Moreover, if $\Phi_3(x, \lambda) = o(e^{\alpha x})$ *as* $x \to \infty$, *then* (7.5.14) *is equivalent to*

$$\Phi_2(x, \lambda) = \Phi_3(x, \lambda) - Y_2(x, -i\alpha) \int_x^\infty Y_3^*(t, -i\alpha)\Phi_3(t, \lambda)\, dt. \quad (7.5.16)$$

2. Let us now consider the boundary-value problem (7.5.C)—(0) defined by (7.5.C) and the boundary condition $Y(0, \lambda) = 0$.

LEMMA 7.5.3: *The matrix* $Y_3(x, \lambda_0)$ *obtained from* $Y_2(x, \lambda_0)$ *by means of the transformation* (7.5.3) *is a matrix eigenfunction of the problem* (7.5.C)—(0) *corresponding to the eigenvalue* λ_0^2 *and*

$$\int_0^\infty Y_3^*(x, \lambda_0)Y_3(x, \lambda_0)\, dx = P.$$

The number $(-i\alpha)^2 = -\alpha^2$ *is not an eigenvalue of this problem.*

Proof: From (7.4.9) and (7.5.5), it follows that

$$\lim_{x \to 0} W\{Y_2^*(x, -i\alpha), Y_2(x, \lambda_0)\} = Y_2^*(0, -i\alpha)Y_2'(0, \lambda_0) =$$
$$= 3\,|a_{12}|^2 m^2 \alpha^{-2}(I-P)P = 0.$$

Therefore the transformation (7.5.3) on $Y_2(x, \lambda_0)$ is expressible as (see Lemma 6.3.1)

$$Y_3(x, \lambda_0) = Y_2(x, \lambda_0) - Y_3(x, -i\alpha) \int_0^x Y_2^*(t, -i\alpha)Y_2(t, \lambda_0)dt, \quad (7.5.17)$$

from which it follows that $Y_3(0, \lambda_0) = 0$. Furthermore, as $x \to \infty$

$$Y_3(x, -i\alpha) = O(e^{-\alpha x}); \quad Y_2(x, -i\alpha) = O(e^{\alpha x});$$
$$Y_2(x, \lambda_0) = O(e^{-|\lambda_0| x})$$

and since $\alpha < |\lambda_0|$, we see from (7.5.17) that $Y_3(x, \lambda_0)$ decays exponentially as $x \to \infty$. Therefore it is a matrix eigenfunction of the problem (7.5.C)—(0).

From (7.5.17), we now obtain

$$\int_0^\infty Y_3^*(x, \lambda_0)Y_3(x, \lambda_0)\, dx = \int_0^\infty Y_2^*(x, \lambda_0)Y_2(x, \lambda_0)\, dx +$$
$$+ \int_0^\infty \int_0^\infty Y_2^*(x, \lambda_0) \left\{ -N(x, y) - N^*(y, x) + \right.$$
$$\left. + \int_0^\infty N^*(t, x)N(t, y)\, dt \right\} Y_2(y, \lambda_0)\, dx\, dy,$$

with

$$N(x, y) = \begin{cases} Y_3(x, -i\alpha)Y_2^*(y, -i\alpha), & x \geq y \\ 0, & x < y. \end{cases}$$

Thus, for $x \geq y$, (7.5.15) and the equality $Y_2(x, -i\alpha)P = 0$ yield

$$-N(x, y) - N^*(y, x) + \int_0^\infty N^*(t, x)N(t, y)\,dt =$$

$$= -Y_3(x, -i\alpha)Y_2^*(y, -i\alpha) +$$

$$+ Y_2(x, -i\alpha)\int_x^\infty Y_3^*(t, -i\alpha)Y_3(t, -i\alpha)\,dt \cdot Y_2^*(y, -i\alpha) =$$

$$= -\xi^{-2}Y_2(x, -i\alpha)PY_2^*(y, -i\alpha) = 0.$$

In a similar way, one can show that this result holds for $x < y$. Therefore

$$\int_0^\infty Y_3^*(x, \lambda_0)Y_3(x, \lambda_0)\,dx = \int_0^\infty Y_2^*(x, \lambda_0)Y_2(x, \lambda_0)\,dx = P$$

(see (7.4.16)), and the first part of the lemma is proved.

Suppose now that $(-i\alpha)^2 = -\alpha^2$ is an eigenvalue of the problem (7.5.C)—(0). Then for $\lambda = -i\alpha$, the equation (7.5.C) has a solution $U_3(x, -i\alpha)$ having the asymptotic behavior

$$U_3(x, -i\alpha) = e^{-\alpha x}[C + o(1)] \qquad (x \to \infty),$$

$$U_3(x, -i\alpha) = xU_3'(0, -i\alpha) + o(x^{1+\theta}) \qquad (x \to 0),$$

where $U_3'(0, -i\alpha)$ and C are non-vanishing matrices.

By the remark of Lemma 6.3.1, we can apply (7.5.16) to $U_3(x, -i\alpha)$ as the result of which we obtain the matrix

$$U_2(x, -i\alpha) = U_3(x, -i\alpha) - Y_2(x, -i\alpha)\int_x^\infty Y_3^*(t, -i\alpha)U_3(t, -i\alpha)\,dt,$$

which is a solution of (7.5.B) for $\lambda = -i\alpha$.

From (7.5.1) and the asymptotic behavior of $U_3(x, -i\alpha)$ at $x = 0$ it follows that

$$U_2(x, -i\alpha) = U_3(x, -i\alpha) - \Phi_2^{(0)}(x, -i\alpha)(I - P)A + O(x^2) \qquad (x \to 0), \tag{7.5.18}$$

$$A = \int_0^\infty Y_3^*(t, -i\alpha)U_3(t, -i\alpha)\,dt.$$

On the other hand, the asymptotic behavior of $U_3(x, -i\alpha)$ at infinity and the formulas (7.5.7) and (7.5.11) show that $U_2(x, -i\alpha) = O(e^{-\alpha x})$ as $x \to \infty$, i.e.,

$$U_2(x, -i\alpha) = E_2(x, -i\alpha)B, \tag{7.5.19}$$

where B is some constant matrix.

Now $B \neq 0$ since $U_2(x, -i\alpha)$ cannot be identically zero. In fact, if $(I - P)A \neq 0$, then for $x = 0$ (7.5.18) yields

$$\begin{aligned} U_2(0, -i\alpha) &= -\Phi_2^{(0)}(0, -i\alpha)(I-P)A = \\ &= -|a_{12}|^2 m^2 \alpha^{-2}(I-P)A \neq 0. \end{aligned}$$

On the other hand, if $(I-P)A = 0$, then $\Phi_2^{(0)}(x, -i\alpha)(I-P)A \equiv 0$, and applying (7.5.18) and the asymptotic representation of $U_3(x, -i\alpha)$ at $x = 0$, we obtain

$$U_2(x, -i\alpha) = xU_3'(0, -i\alpha) + o(x^{1+\theta}) \qquad (x \to 0).$$

Thus again $U_2(x, -i\alpha) \not\equiv 0$ since $U_3'(0, -i\alpha) \neq 0$.

From (7.5.18) and Lemma 7.4.2, it then follows that

$$\begin{aligned} &\lim_{x\to 0} W\{U_2^*(x, -i\alpha), \Phi_2^{(0)}(x, -i\alpha)\} = \\ &= \lim_{x\to 0} W\{U_3^*(x, -i\alpha), \Phi_2^{(0)}(x, -i\alpha)\} = \\ &= -U_3^{*\prime}(0, -i\alpha)\Phi_2^{(0)}(0, -i\alpha). \end{aligned}$$

Hence,

$$\begin{aligned} &W\{U_2^*(x, -i\alpha), \Phi_2^{(0)}(x, -i\alpha)\} \equiv \\ &\equiv -|a_{12}|^2 m^2 \alpha^{-2} U_3^{*\prime}(0, -i\alpha)(I-P), \end{aligned} \tag{7.5.20}$$

since the Wronskian is independent of x.

The Wronskian

$$W\{U_3^*(x, -i\alpha), Y_3(x, -i\alpha)\}$$

is also independent of x and since both of the solutions $U_3(x, -i\alpha)$ and $Y_3(x, -i\alpha)$ decrease exponentially as $x \to \infty$, it is identically equal to zero. Therefore

$$0 = \lim_{x\to 0} W\{U_3^*(x, -i\alpha), Y_3(x, -i\alpha)\} = -U_3^{*\prime}(0, -i\alpha)Y_3(0, -i\alpha)$$

and by (7.5.2) and (7.5.5),

$$-|a_{12}|^2 m^2 \alpha^{-2} \xi^{-2} U_3^{*\prime}(0, -i\alpha)(I-P) = 0.$$

This result in conjunction with (7.5.20) then yields

$$W\{U_2^*(x, -i\alpha), \Phi_2^{(0)}(x, -i\alpha)\} \equiv 0. \tag{7.5.21}$$

Now the columns of $\Phi_2^{(0)}(x, -i\alpha)$ give two linearly independent solutions of equation (7.4.B) for $\lambda = -i\alpha$. Therefore, a matrix solution $\Psi_2(x, -i\alpha)$ of equation (7.4.B) exists for $\lambda = -i\alpha$ such that

$$W\{\Psi_2^*(x, -i\alpha), \Phi_2^{(0)}(x, -i\alpha)\} \equiv I,$$

and the matrices $\Phi_2^{(0)}(x, -i\alpha)$ and $\Psi_2(x, -i\alpha)$ comprise a fundamental system of solutions of this equation. As a result,

$$U_2(x, -i\alpha) = \Phi_2^{(0)}(x, -i\alpha)B_1 + \Psi_2'(x, -i\alpha)B_2,$$

where B_1 and B_2 are constant matrices. From this, Lemma 7.4.2, and the identity (7.5.21), if follows that

$$0 = W\{U_2^*(x, -i\alpha), \Phi_2^{(0)}(x, -i\alpha)\} =$$
$$= B_2^* W\{\Psi_2^*(x, -i\alpha), \Phi_2^{(0)}(x, -i\alpha)\} = B_2^*,$$

or $B_2 = 0$. Hence,

$$U_2(x, -i\alpha) = \Phi_2^{(0)}(x, -i\alpha)B_1,$$

and by (7.5.19),

$$E_2(x, -i\alpha)B = \Phi_2^{(0)}(x, -i\alpha)B_1 \qquad (B \neq 0).$$

Since the transformations of the preceding sections do not effect the asymptotic behavior of the solutions at infinity, this last result implies that

$$E(x, -i\alpha)B = \Phi^{(0)}(x, -i\alpha)B_1 \qquad (B \neq 0).$$

Thus, $E(x, -i\alpha)\ B$ is a non-trivial solution of equation (7.1.0) for $\lambda = -i\alpha$ which vanishes at $x = 0$ and which decreases exponentially as $x \to \infty$. This means that $(-i\alpha)^2 = -\alpha^2$ is an eigenvalue of the boundary-value problem (7.1.0)—(0) regardless of the choice of α. The resulting contradiction shows that $(-i\alpha)^2 = -\alpha^2$ cannot be an eigenvalue of the boundary-value problem (7.5.C)—(0), q.e.d.

3. Since the equation (7.5.C) is non-singular, the problem (7.5.C)—(0) has a continuous pectrum on the positive axis and, in addition to the eigenvalue λ_0^2 and corresponding eigenfunction $Y_3(x, \lambda_0)$, possibly a finite number of other negative eigenvalues $\lambda_1^2, \lambda_2^2, \ldots, \lambda_p^2$.

Let the matrices $\hat{U}_3(x, \lambda)$ and $\hat{U}_3(x, \lambda_k)$ $(k = 0,1, \ldots, p)$ form a complete set of eigenfunctions of this problem for which Parseval's equality

$$\int_0^\infty \Phi(x)\Psi^*(x)\,dx = \sum_{k=0}^{p} \hat{U}_3(\Phi;\ \lambda_k)\hat{U}_3^*(\Psi;\ \lambda_k) + \frac{1}{2\pi}\int_0^\infty \hat{U}_3(\Phi;\ \lambda)\hat{U}_3^*(\Psi;\ \lambda)\,d\lambda \tag{7.5.22}$$

holds, and such that at infinity

$$\begin{aligned} \hat{U}_3(x,\lambda) &= e^{i\lambda x}I - e^{-i\lambda x}\hat{S}(-\lambda) + o(1) \qquad (\lambda > 0); \\ \hat{U}_3(x,\lambda_k) &= e^{-|\lambda_k|x}[\hat{M}_k + o(1)] \qquad (k = 0, 1, 2, \dots, p). \end{aligned} \tag{7.5.23}$$

Here, $\hat{S}(\lambda)$, λ_k^2, and $\hat{M}_k$ comprise the scattering data of the problem in question.

From Lemma 7.5.3, if follows that $\hat{U}_3(x,\lambda_0) = Y_3(x,\lambda_0)\,U$, where U is some fixed unitary matrix.

We now apply to $\hat{U}_3(x,\lambda)$ the transformation inverse to (7.5.3) which by Lemma 7.5.2 can be expressed as

$$\hat{U}_2(x,\lambda) = \hat{U}_3(x,\lambda) - Y_2(x,\ -i\alpha)\int_x^\infty Y_3^*(t,\ -i\alpha)\hat{U}_3(t,\lambda)\,dt. \tag{7.5.24}$$

But $\hat{U}_2(x,\lambda)$ satisfies equation (7.4.B), and by Theorem 6.3.1 and formula (7.5.15) of Lemma 7.5.2 we have

$$\int_0^\infty \Phi(x)\Phi^*(x)\,dx = \sum_{k=0}^{p} \hat{U}_2(\Phi;\ \lambda_k)\hat{U}_2^*(\Phi;\ \lambda_k) + \frac{1}{2\pi}\int_0^\infty \hat{U}_2(\Phi;\ \lambda)\hat{U}_2^*(\Phi;\ \lambda)\,d\lambda + \xi^{-2}Y_2(\Phi;\ -i\alpha)PY_2^*(\Phi;\ -i\alpha),$$

for any matrix $\Phi(x)$ in $L^2(0,\infty)$ vanishing in the neighborhood of zero and infinity. Since $Y_2(x,-i\alpha)\,(I-P) = Y_2(x,-i\alpha)$, this implies that $Y_2(\Phi;-i\alpha)\,P = 0$, and we obtain the Parseval equality

$$\int_0^\infty \Phi(x)\Phi^*(x)\,dx = \sum_{k=0}^{p} \hat{U}_2(\Phi;\ \lambda_k)\hat{U}_2^*(\Phi;\ \lambda_k) + \frac{1}{2\pi}\int_0^\infty \hat{U}_2(\Phi;\ \lambda)\hat{U}_2^*(\Phi;\ \lambda)\,d\lambda.$$

This result can now be extended in the usual way to any matrix $\Phi(x)$ with elements in $L^2(0, \infty)$, where on the basis of the above discussion,

$$\hat{U}_2(x, \lambda_0) = Y_2(x, \lambda_0)U.$$

We note that the transformation introduced in this section effects the asymptotic form of the solutions at infinity.

LEMMA 7.5.4: *As* $x \to \infty$,

$$\hat{U}_2(x, \lambda) = e^{i\lambda x}T(-\lambda) - e^{-i\lambda x}T(\lambda)\hat{S}(-\lambda) + o(1) \qquad (\lambda > 0),$$

$$\hat{U}_2(x, \lambda_k) = e^{-|\lambda_k|x}[T(\lambda_k)\hat{M}_k + o(1)] \qquad (k = 0, 1, 2, \ldots, p),$$

where

$$T(\lambda) = I - 2\alpha(\alpha + i\lambda)^{-1}Q \tag{7.5.25}$$

and Q *is some projecting Hermitian matrix of rank one.* $T(\lambda)$ *is nonsingular for all values of* $\lambda \neq \pm i\alpha$ *and unitary for all real* λ.

Proof: By means of (7.5.7), (7.5.11), and (7.5.23), we find from (7.5.24) that as $x \to \infty$,

$$\hat{U}_2(x, \lambda) = e^{i\lambda x}I - e^{-i\lambda x}\hat{S}(-\lambda) + o(1) -$$

$$-2\alpha c^{-1}e^{\alpha x}[C_2 + o(1)]\,(I-P)\int_x^\infty e^{-\alpha t}[C_2^* + o(1)]\,[e^{i\lambda t}I -$$

$$-e^{-i\lambda t}\hat{S}(-\lambda) + o(1)]\,dt = e^{i\lambda x}[I - 2\alpha c^{-1}(\alpha - i\lambda)^{-1}C_2(I-P)C_2^*] -$$

$$-e^{-i\lambda x}[I - 2\alpha c^{-1}(\alpha + i\lambda)^{-1}C_2(I-P)C_2^*]\hat{S}(-\lambda) + o(1)$$

for $\lambda > 0$. Hence,

$$\hat{U}_2(x, \lambda) = e^{i\lambda x}T(-\lambda) - e^{-i\lambda x}T(\lambda)\hat{S}(-\lambda) + o(1) \qquad (x \to \infty),$$

where

$$T(\lambda) = I - 2\alpha c^{-1}(\alpha + i\lambda)^{-1}C_2(I-P)C_2^*.$$

In a similar way, for $\lambda = \lambda_k$ ($k = 0,1,2, \ldots, p$) it can be shown that

$$\hat{U}_2(x, \lambda_k) = e^{-|\lambda_k|x}[T(\lambda_k)\hat{M}_k + o(1)] \qquad (x \to \infty).$$

Defining

$$Q = c^{-1}C_2(I-P)C_2^*. \tag{7.5.26}$$

we obtain the expression (7.5.25) for $T(\lambda)$. From this definition it is immediately apparent that Q is a Hermitian matrix of rank one.

Moreover, from (7.5.10), it follows that

$$Q^2 = c^{-2}C_2(I-P)C_2^*C_2(I-P)C_2^* = c^{-2}C_2c(I-P)C_2^* =$$
$$= c^{-1}C_2(I-P)C_2^* = Q,$$

i.e., Q is a projection.

Now for $\lambda \neq \pm i\alpha$

$$T(\lambda)T(-\lambda) = [I-2\alpha(\alpha+i\lambda)^{-1}Q][I-2\alpha(\alpha-i\lambda)^{-1}Q] =$$
$$= I-2\alpha(\alpha+i\lambda)^{-1}Q-2\alpha(\alpha-i\lambda)^{-1}Q+4\alpha^2(\alpha^2+\lambda^2)^{-1}Q = I$$

so that $T(\lambda)$ is a non-singular matrix for these values of λ. Noting further that by (7.5.25)

$$T(-\lambda) = T^*(\lambda)$$

for real values of λ, we conclude that $T(\lambda)\,T^*(\lambda) = I$. This completes the proof of the lemma.

4. Upon applying the transformation (7.4.14) to $\hat{U}_2(x, \lambda)$, we obtain a solution of equation (7.3.A). In order to write this transformation in integral form, we must show that the appropriate Wronskian is zero.

From (7.4.9) it follows that

$$\lim_{x\to 0} W\{Y_2^*(x, \lambda_0),\ \hat{U}_2(x, \lambda)\} = -3P\hat{U}_2(0, \lambda). \tag{7.5.27}$$

On the other hand, by (7.5.24),

$$\hat{U}_2(x, \lambda) = \hat{U}_3(x, \lambda)-Y_2(x,\ -i\alpha)A(\lambda)+O(x^2) \qquad (x\to 0), \tag{7.5.28}$$

where

$$A(\lambda) = \int_0^\infty Y_3^*(t,\ -i\alpha)\hat{U}_3(t, \lambda)\,dt =$$
$$= -(\lambda^2+\alpha^2)^{-1}W\{Y_3^*(x,\ -i\alpha),\ \hat{U}_3(x, \lambda)\}\Big|_0^\infty =$$
$$= (\lambda^2+\alpha^2)^{-1}Y_3^*(0,\ -i\alpha)\hat{U}_3'(0, \lambda). \tag{7.5.29}$$

Hence, by (7.5.5),

$$\hat{U}_2(0, \lambda) = -|a_{12}|^2m^2\alpha^{-2}(I-P)A(\lambda).$$

Substituting this expression for $\hat{U}_2(0, \lambda)$ in (7.5.27), we obtain

$$\lim_{x\to 0} W\{Y_2^*(x, \lambda_0),\ \hat{U}_2(x, \lambda)\}=0.$$

Therefore, by virtue of Lemma 6.3.1 and its remark, the transformation (7.4.14) applied to $\hat{U}_2(x, \lambda)$ may be written in the form

$$\hat{U}_1(x, \lambda) = \hat{U}_2(x, \lambda) - Y_1(x, \lambda_0) \int_0^x Y_2^*(t, \lambda_0)\hat{U}_2(t, \lambda)\, dt. \quad (7.5.30)$$

According to the results of the preceding section, this transformation takes $\hat{U}_2(x, \lambda)$ for $\lambda \neq \lambda_0$ into a solution $\hat{U}_1(x, \lambda)$ of the equation (7.3.A) without effecting its asymptotic behavior at infinity and annihilates the matrix $\hat{U}_2(x, \lambda_0) = Y_2(x, \lambda_0)\, U$, i.e., $\hat{U}_1(x, \lambda_0) \equiv 0$.

Upon applying Theorem 6.3.1 to the transformation (7.5.30) and taking into consideration that both $Y_1(x, \lambda_0)\,(I-P)$ and $\hat{U}_1(x, \lambda_0)$ are identically zero, we obtain Parseval's equality

$$\int_0^\infty \Phi(x)\Phi^*(x)\, dx = \sum_{k=1}^{p} \hat{U}_1(\Phi;\ \lambda_k)\hat{U}_1^*(\Phi;\ \lambda_k) +$$

$$+\frac{1}{2\pi}\int_0^\infty \hat{U}_1(\Phi;\ \lambda)\hat{U}_1^*(\Phi;\ \lambda)\, d\lambda. \quad (7.5.31)$$

This result can be extended in the usual way so as to hold for any matrix $\Phi(x)$ with elements in $L^2(0, \infty)$.

5. Before considering the final transformation that will take us back to the equation (7.1.0), we must ascertain the behavior of $\hat{U}_1(x, \lambda)$ in the vicinity of zero.

The formulas (7.5.28) and (7.5.30) yield

$$\hat{U}_1(x, \lambda) = \hat{U}_3(x, \lambda) - Y_1(x, \lambda_0) \int_0^x Y_2^*(t, \lambda_0)\hat{U}_3(t, \lambda)\, dt$$

$$-\left[Y_2(x, -i\alpha) - Y_1(x, \lambda_0) \int_0^x Y_2^*(t, \lambda_0) Y_2(t, -i\alpha) dt\right] A(\lambda) +$$

$$+ O(x^2) - Y_1(x, \lambda_0) \int_0^x Y_2^*(t, \lambda_0) O(t^2)\, dt \qquad (x \to 0).$$

Hence, the asymptotic relations (Cf. (7.4.4) and (7.4.9))

$$\begin{aligned} \hat{U}_3(x, \lambda) &= x\hat{U}_3'(0, \lambda) + o(x^{1+\theta}), \\ Y_2(x, \lambda_0) &= 3xP + o(x^{1+\theta})P, \qquad (x \to 0) \\ Y_1(x, \lambda_0) &= x^{-2}P + + o(x^{-2+\theta})P \end{aligned}$$

can be used to show that

$$\hat{U}_1(x, \lambda) = x(I-P)\hat{U}_3'(0, \lambda) - Y_1(x, -i\alpha)A(\lambda) + o(x^{1+\theta}) \qquad (x \to 0), \tag{7.5.32}$$

where

$$Y_1(x, -i\alpha) = Y_2(x, -i\alpha) - Y_1(x, \lambda_0)\int_0^x Y_2^*(t, \lambda_0)Y_2(t, -i\alpha)\,dt.$$

From this and the fact that

$$\lim_{x\to 0} W\{Y_2^*(x, \lambda_0), Y_2(x, -i\alpha)\} = -Y_2^{*\prime}(0, \lambda_0)Y_2(0, -i\alpha) = 0$$

(see (7.4.9) and (7.5.5)), we conclude by Lemma 6.3.1 that $Y_1(x, -i\alpha)$ is the image of $Y_2(x, -i\alpha)$ under the transformation (7.4.14). Therefore, by virtue of (7.5.1)

$$Y_1(x, -i\alpha) = \Phi_1^{(0)}(x, -i\alpha)(I-P).$$

On the basis of Lemma 7.3.2, we thus have

$$Y_1(x, -i\alpha) = \Phi^{(0)}(x, -i\alpha)(I-P) + \\ + Y_1(x)B_0(-i\alpha)(I-P) + O(x^2) \qquad (x \to 0),$$

or

$$Y_1(x, -i\alpha) = x(I-P) + Y_1(x)B_0(-i\alpha)(I-P) + o(x^{1+\theta}) \qquad (x \to 0).$$

Substituting this expression and the expression (7.5.29) for $A(\lambda)$ in (7.5.32), we obtain

$$\hat{U}_1(x, \lambda) = \{(I-P)x - [xI + Y_1(x)B_0(-i\alpha)]\times \\ \times (I-P)Y_3^*(0, -i\alpha)(\lambda^2+\alpha^2)^{-1}\}\hat{U}_3'(0, \lambda) + o(x^{1+\theta}) \qquad (x \to 0).$$

And since by (7.5.2), (7.5.5), and (7.5.13)

$$Y_3(0, -i\lambda) = |a_{12}|^2 m^2\alpha^{-2}\xi^{-2}(I-P) = \alpha^2(I-P), \tag{7.5.33}$$

this yields

$$\hat{U}_1(x, \lambda) = [\lambda^2 xI - Y_1(x)B_0(-i\alpha)\alpha^2](\lambda^2+\alpha^2)^{-1}(I-P)\hat{U}_3'(0, \lambda) + \\ + o(x^{1+\theta}) \tag{7.5.34}$$

as $x \to 0$.

Setting $x = 0$ in this, we find that

$$\hat{U}_1(0, \lambda) = -Y_1(0)B_0(-i\alpha)\alpha^2(\lambda^2+\alpha^2)^{-1}(I-P)U_3'(0, \lambda),$$

and since $Y_1(0)\, B_0(-i\alpha) = |a_{12}|^2\, m^2\, \alpha^{-2}(I-P)$ (see Lemma 7.3.2), it follows that

$$\hat{U}_1(0,\lambda) = -|a_{12}|^2 m^2(\lambda^2+\alpha^2)^{-1}(I-P)\hat{U}_3'(0,\lambda).$$

Hence,

$$(\lambda^2+\alpha^2)^{-1}(I-P)\hat{U}_3'(0,\lambda) = -|a_{12}|^{-2}m^{-2}\hat{U}_1(0,\lambda),$$

and thus (7.5.34) can be expressed in the form

$$\hat{U}_1(x,\lambda) = -|a_{12}|^{-2}m^{-2}[\lambda^2 xI - Y_1(x)B_0(-i\alpha)\alpha^2]\hat{U}_1(0,\lambda) + \\ + o(x^{1+\theta}) \qquad (x\to 0).$$

But this result and the identity $W\{Y_1^*(x), Y_1(x)\} \equiv 0$ show that

$$\lim_{x\to 0} W\{Y_1^*(x),\ \hat{U}_1(x,\lambda)\} = -|a_{12}|^{-2}m^{-2}\lambda^2 Y_1^*(0)\hat{U}_1(0,\lambda).$$

Therefore, on the basis of Lemma 6.3.1, we can write the transformation (7.3.17) on $\hat{U}_1(x,\lambda)$ in the form

$$\hat{U}(x,\lambda) = \hat{U}_1(x,\lambda) + Y(x)\int_0^x Y_1^*(t)\hat{U}_1(t,\lambda)\,dt + Y(x)B(\lambda),$$

$$B(\lambda) = |a_{12}|^{-2}m^{-2}Y_1^*(0)\hat{U}_1(0,\lambda). \tag{7.5.35}$$

According to the results of § 3 of this chapter, $\hat{U}(x,\lambda)$ satisfies the given equation (7.1.0) and has the same asymptotic behavior at infinity as does $\hat{U}_1(x,\lambda)$ and therefore as does $\hat{U}_2(x,\lambda)$. Furthermore, from the formula (7.5.35) for $x = 0$, we obtain

$$\hat{U}(0,\lambda) = [I + |a_{12}|^{-2}m^{-2}Y(0)Y_1^*(0)]\hat{U}_1(0,\lambda) = P\hat{U}_1(0,\lambda) = 0$$

(see (7.3.12) and (7.3.14)) or in other words, $\hat{U}(x,\lambda)$ satisfies the boundary condition $\hat{U}(0,\lambda) = 0$.

We next show that Parseval's equality holds for the family of matrices $\hat{U}(x,\lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1,2,\ldots,p$). The expressions (7.5.30) and (7.5.24) give

$$\hat{U}_1(0,\lambda) = \hat{U}_2(0,\lambda) = \\ = -Y_2(0,-i\alpha)\int_0^\infty Y_3^*(t,-i\alpha)\hat{U}_3(t,\lambda)\,dt, \tag{7.5.36}$$

and by applying the Parseval equality (7.5.22), we obtain

$$\sum_{k=0}^{p} \hat{U}_1(0, \lambda_k)\hat{U}_1^*(0, \lambda_k) + \frac{1}{2\pi}\int_0^\infty \hat{U}_1(0, \lambda)\hat{U}_1^*(0, \lambda)\, d\lambda =$$

$$= Y_2(0, -i\alpha)\int_0^\infty Y_3^*(t, -i\alpha)Y_3(t, -i\alpha)\, dt\, Y_2^*(0, -i\alpha).$$

But by (7.5.5) and (7.5.15),

$$Y_2(0, -i\alpha)\int_0^\infty Y_3^*(t, -i\alpha)Y_3(t, -i\alpha)\, dt =$$

$$= Y_2(0, -i\alpha)\left[\xi^{-2}P + \int_0^\infty Y_3^*(t, -i\alpha)Y_3(t, -i\alpha)\, dt\right] = Y_3(0, -i\alpha),$$

and as previously shown, $\hat{U}_1(x, \lambda_0) \equiv 0$. Thus

$$\sum_{k=1}^{p} \hat{U}_1(0, \lambda_k)\hat{U}_1^*(0, \lambda_k) + \frac{1}{2\pi}\int_0^\infty \hat{U}_1(0, \lambda)\hat{U}_1^*(0, \lambda)\, d\lambda =$$

$$= Y_3(0, -i\alpha)Y_2^*(0, -i\alpha) = |a_{12}|^2 m^2 (I-P)$$

(see (7.5.5) and (7.5.33)) and, therefore, the matrix $B(\lambda)$ appearing in (7.5.35) satisfies the relation

$$\sum_{k=1}^{p} B(\lambda_k)B^*(\lambda_k) + \frac{1}{2\pi}\int_0^\infty B(\lambda)B^*(\lambda)\, d\lambda =$$

$$= |a_{12}|^{-2} m^{-2} Y_1^*(0)(I-P)Y_1(0) = m^2 P. \tag{7.5.37}$$

This shows that hypothesis *(1)'* of Theorem 6.3.1 holds for the transformation (7.5.35).

Let us verify that the second hypothesis of Theorem 6.3.1 also holds. Let $H(t)$ denote the following matrix step-function:

$$H(t) = \begin{cases} \dfrac{1}{h} I, & 0 \le t \le h \\ 0, & t > h. \end{cases}$$

From the Parseval equality (7.5.31) it follows that

$$\sum_{k=1}^{p} \hat{U}_1(H;\ \lambda_k)\hat{U}_1^*(\Psi;\ \lambda_k)+\frac{1}{2\pi}\int_0^\infty \hat{U}_1(H;\ \lambda)\hat{U}_1^*(\Psi;\ \lambda)\,d\lambda = 0, \tag{7.5.38}$$

provided that $\Psi(x)$ is equal to zero for $x < h$.

Now

$$\lim_{h\to 0} \hat{U}_1(H;\ \lambda)=\lim_{h\to 0}\frac{1}{h}\int_0^h \hat{U}_1(x,\ \lambda)\,dx=\hat{U}_1(0,\ \lambda)$$

because of the continuity of $\hat{U}_1(x, \lambda)$ at $x = 0$ (see (7.5.34)). Furthermore, since

$$\sup_{\substack{0\le x<\infty\\ 0<\lambda<\infty}} |\hat{U}_3(x,\ \lambda)| <\infty,$$

it is easy to show by the use of (7.5.24) and (7.5.30), that

$$\sup_{\substack{0\le x<\infty\\ 0<\lambda<\infty}} |\hat{U}_1(x,\ \lambda)| <\infty. \tag{7.5.39}$$

Hence, as $h \to 0$, $\hat{U}_1(H; \lambda)$ converges boundedly to $\hat{U}_1(0, \lambda)$ $(0 < \lambda < \infty)$. Thus, if $\Psi(t)$ is twice continuously differentiable and equal to zero in the neighborhood of zero and infinity (in this event, $\hat{U}_1(\Psi; \lambda) = O(\lambda^{-2})$ as $\lambda \to \infty$), we may then let $h \to 0$ underneath the integral sign in (7.5.38). As a result we obtain

$$\sum_{k=1}^{p} \hat{U}_1(0,\ \lambda_k)\hat{U}_1^*(\Psi;\ \lambda_k)+\frac{1}{2\pi}\int_0^\infty \hat{U}_1(0,\ \lambda)\hat{U}_1^*(\Psi;\ \lambda)\,d\lambda = 0. \tag{7.5.40}$$

According to (7.5.36), the elements of $\hat{U}_1(0, \lambda)$ belong to $L^2(0, \infty)$, whereas the corresponding elements of the matrices $\Psi(t)$ (with the properties specified above) form a set which is everywhere dense in $L^2(0, \infty)$. Therefore (7.5.40) holds for every matrix with elements in $L^2(0, \infty)$. This shows that hypothesis (*2*) of Theorem 6.3.1 is satisfied.

This theorem and the relations (7.5.31), (7.5.35), (7.3.18′), and (7.5.37) thus yield

$$\int_0^\infty \Phi(x)\Phi^*(x)dx = \sum_{k=1}^{p} \hat{U}(\Phi;\ \lambda_k)\hat{U}^*(\Phi;\ \lambda_k) + \\ + \frac{1}{2\pi}\int_0^\infty \hat{U}(\Phi;\ \lambda)\hat{U}^*(\Phi;\ \lambda)\,d\lambda - Y(\Phi)\,(I-P)Y^*(\Phi).$$

But since $Y(x)(I-P) \equiv 0$, $Y(\Phi)(1-P) = 0$ and hence it follows that

$$\int_0^\infty \Phi(x)\Phi^*(x)\,dx = \sum_{k=1}^{p} \hat{U}(\Phi;\ \lambda_k)\hat{U}^*(\Phi;\ \lambda_k) + \\ + \frac{1}{2\pi}\int_0^\infty \hat{U}(\Phi;\ \lambda)\hat{U}^*(\Phi;\ \lambda)\,d\lambda. \tag{7.5.41}$$

We may take this result for granted for any matrix $\Phi(x)$ with elements in $L^2(0, \infty)$.

6. As already indicated above, the matrices $\hat{U}(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1, 2, \ldots, p$) have the same asymptotic behavior at infinity as $\hat{U}_2(x, \lambda)$. With the idea of applying Lemma 7.5.4, we now define

$$U(x,\ \lambda) = \hat{U}(x,\ \lambda)T(\lambda) \qquad (\lambda > 0),$$

where $T(\lambda)$ is the unitary matrix given by (7.5.25). Next, making use of the polar decomposition[5]

$$T(\lambda_k)\hat{M}_k = M_k U_k \qquad (k = 0,\ 1,\ 2, \ldots, p),$$

where M_k is a non-negative Hermitian matrix and U_k a unitary matrix, we let

$$U(x,\ \lambda_k) = \hat{U}(x,\ \lambda_k)U_k^* \qquad (k = 1,\ 2, \ldots, p).$$

Since $U(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1, 2, \ldots, p$) is the product of $\hat{U}(x, \lambda)$ and a unitary matrix not depending on x, it also satisfies

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[5] See, for example, Maltsev, ***Fundamentals of Linear Algebra***, Chap. VIII, § 2, 1948 (in Russian); or Halmos, ***Finite Dimensional Vector Spaces***, 1958, pg. 169, Van Nostrand.

7.1.0) and so according to (7.5.41), the Parseval equality

$$\int_0^\infty \Phi(x)\Phi^*(x)dx = \sum_{k=1}^{p} U(\Phi;\ \lambda_k)U^*(\Phi;\ \lambda_k) + $$

$$+\frac{1}{2\pi}\int_0^\infty U(\Phi;\ \lambda)U^*(\Phi;\ \lambda)\,d\lambda \tag{7.5.42}$$

holds. Moreover, by the definition of $U(x, \lambda)$ and Lemma 7.5.4,

$$\left.\begin{aligned} U(x,\ \lambda) &= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0),\\ U(x,\ \lambda_k) &= e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 1,\ 2, \ldots, p). \end{aligned}\right\} \quad (x \to \infty)$$

Here,

$$S(-\lambda) = T(\lambda)\hat{S}(-\lambda)T(\lambda),$$

$$M_k = T(\lambda_k)\hat{M}_k U_k^*, \tag{7.5.43}$$

and as can easily be shown, $S(\lambda) = S^*(-\lambda)$ is a unitary matrix and the non-negative Hermitian matrix M_k has the same rank as $\hat{M}_k$.

Thus we have proved

THEOREM 7.5.1: *If* $a_{22} = 0$ *and* $a_{12} \neq 0$ (*i.e.,* $Y(0)$ *is bounded and* $Y(0) \neq 0$), *then the problem* (7.1.0)—(0) *has a finite number of eigenvalues* $\lambda_k^2 < 0$ ($k = 1,\ 2, \ldots, p$) *and a continuous spectrum over the entire positive axis. A complete set of normalized matrix eigenfunctions* $U(x, \lambda)$ *of this problem exists for which Parseval's equality* (7.5.42) *holds and such that*

$$U(x,\ \lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0),$$

$$U(x,\ \lambda_k) = e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 1,\ 2, \ldots, p)$$

as $x \to \infty$, *where* $S(\lambda) = S^*(-\lambda)$ *is a unitary matrix* (*the scattering matrix*) *and* M_k *are non-negative Hermitian matrices* (*the normalization matrices*).

REMARK. The exact same reasoning that was applied above to the matrices $\hat{U}(x, \lambda)$ and $U(x, \lambda)$ may be used to go from $\hat{U}_i(x, \lambda)$ ($i = 1,2$) to the matrices

$$U_i(x,\ \lambda) = \hat{U}_i(x,\ \lambda)T(\lambda) \qquad (\lambda > 0);$$

$$U_i(x,\ \lambda_k) = \hat{U}_i(x,\ \lambda_k)U_k^* \qquad (k = 1,\ 2, \ldots, p).$$

The matrices $U_i(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1, 2, \ldots, p$) satisfy the same equations as $\hat{U}_i(x, \lambda)$ and lead to Parseval's equality; moreover, for $\lambda \neq \lambda_0$, $U_i(x, \lambda)$ has the same asymptotic form at infinity as does $U(x, \lambda)$. For $\lambda = \lambda_0$, an argument similar to the above shows that

$$U_1(x, \lambda_0) = \hat{U}_1(x, \lambda_0) \equiv 0$$

and

$$U_2(x, \lambda_0) = e^{-|\lambda_0|x}[M_0 + o(1)] \qquad (x \to \infty),$$

where

$$M_0 = T(\lambda_0)\hat{M}_0 U_0^*.$$

§ 6. Properties of the Scattering Data (the Case $a_{22} = 0$, $a_{12} \neq 0$)

In this section we shall study the properties of the scattering data of the boundary-value problem (7.1.0)—(0) for the case $a_{22} = 0$, $a_{12} \neq 0$.

If $a_{22} \neq 0$, or $a_{22} = a_{12} = 0$, then by the corollaries to Theorems 7.3.1 and 7.4.1, $I—S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$, such that

$$F_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [I - S(\lambda)] e^{i\lambda t} \, d\lambda$$

is summable over the entire real axis. Its derivative $F_s'(t)$ exists for all positive values of t and

$$\int_0^{\infty} t^{1+\theta} \, | F_s'(t) | \, dt < \infty \qquad (| \theta | < \varepsilon). \tag{7.6.1}$$

We now show that $S(\lambda)$ possesses these properties even if $a_{12} \neq 0$ when $a_{22} = 0$.

According to (7.5.43), $S(\lambda)$ can be expressed in this case in terms of the scattering matrix $\hat{S}(\lambda)$ of the boundary-value problem (7.5.C)—(0) as follows:

$$S(\lambda) = T(-\lambda)\hat{S}(\lambda)T(-\lambda).$$

Therefore, using the expression (7.5.25) for $T(\lambda)$, we obtain

$$\begin{aligned} I - S(\lambda) &= I - [I - 2\alpha(\alpha - i\lambda)^{-1}Q]\hat{S}(\lambda)[I - 2\alpha(\alpha - i\lambda)^{-1}Q] = \\ &= [I - \hat{S}(\lambda)] - 2\alpha(\alpha - i\lambda)^{-1}\{Q[I - \hat{S}(\lambda)] + [I - \hat{S}(\lambda)]Q\} + \\ &+ 4\alpha^2(\alpha - i\lambda)^{-2}Q[I - \hat{S}(\lambda)]Q + 4\alpha(\alpha - i\lambda)^{-1}[1 - \alpha(\alpha - i\lambda)^{-1}]Q. \end{aligned} \tag{7.6.2}$$

The problem (7.5.C)—(0) is not singular and

$$\int_0^\infty x^{1+\theta} \mid V_3(x) \mid dx < \infty$$

for $\mid \theta \mid < \varepsilon$. Therefore $I - \hat{S}(\lambda)$ is the Fourier transform of a Hermitian matrix $\hat{F}_s(t)$ summable over the entire real axis (see Theorem 5.6.2) and such that $\hat{F}_s'(t)$ satisfies (7.6.1). Since

$$(\alpha - i\lambda)^{-1} = \int_{-\infty}^0 e^{\alpha t} e^{-i\lambda t}\, dt, \quad -(\alpha - i\lambda)^{-2} = \int_{-\infty}^0 t e^{\alpha t} e^{-i\lambda t}\, dt,$$

by applying the convolution theorem,[6] we find from (7.6.2) that $I - S(\lambda)$ is the Fourier transform of

$$F_s(t) = \begin{cases} \hat{F}_s(t) - 2\alpha\Phi(t) - 4\alpha^2\Psi(t) & \text{for} \quad t > 0 \\ \hat{F}_s(t) - 2\alpha\Phi(t) - 4\alpha^2\Psi(t) + 4\alpha(1+\alpha t)e^{\alpha t}Q & \text{for} \quad t < 0, \end{cases} \tag{7.6.3}$$

where

$$\Phi(t) = \int_{-\infty}^0 e^{\alpha\xi}[Q\hat{F}_s(t-\xi) + \hat{F}_s(t-\xi)Q]\, d\xi =$$
$$= e^{\alpha t} \int_t^\infty e^{-\alpha\eta}[Q\hat{F}_s(\eta) + \hat{F}_s(\eta)Q]\, d\eta, \tag{7.6.4}$$

$$\Psi(t) = \int_{-\infty}^0 \xi e^{\alpha\xi} Q\hat{F}(t-\xi)Q\, d\xi = e^{\alpha t} \int_t^\infty (t-\eta)e^{-\alpha\eta} Q\hat{F}_s(\eta)Q\, d\eta.$$

From this result, the properties of $\hat{F}_s(t)$, and the convolution theorem, it follows that $F_s(t)$ is Hermitian, is summable over the entire real axis, and has a derivative for all positive values of t. The fact that $F_s'(t)$ satisfies (7.6.1) may be verified directly using (7.6.3) and (7.6.4). However, this is unnecessary since this inequality will be derived below in the course of the proof of Lemma 7.6.1.

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[6] See footnote 1 of Chapter III.

We can deduce other properties of the scattering data by examining the homogeneous integral equations

$$x(t)+\int_0^\infty x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty, \tag{7.6.5}$$

$$-y(t)+\int_{-\infty}^0 y(\xi)F_s(t+\xi)\,d\xi = 0, \qquad -\infty < t \leq 0, \tag{7.6.6}$$

$$z(t)+\int_0^\infty z(\xi)F_s(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty, \tag{7.6.7}$$

where

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k|t}+F_s(t).$$

Let n_1, n_2, and n_3 denote respectively the number of linearly independent vector solutions of these equations $[x(t)$ and $z(t)$ in $L^2_{(2)}(0, \infty)$ and $y(t)$ in $L^2_{(2)}(-\infty, 0)]$, and let r be the sum of the ranks of the p matrices M_k.

LEMMA 7.6.1: *If* $a_{22} = 0$, *but* $a_{12} \neq 0$, *then* $n_1 = 1$, $n_3 = n_2+r$, *and* n_2 *is zero or one. Moreover, if* $\tilde{x}(\lambda)$ *is the Fourier transform of the single non-trivial solution of* (7.6.5), *then*

$$\lim_{\lambda\to\infty} \lambda^2\tilde{x}(\lambda)\,(I-P) = 0.$$

Proof: According to the remark of Theorem 7.5.1, $U_2(x, \lambda)$ $(\lambda > 0;$ $\lambda = \lambda_k$, $k = 0, 1, \ldots, p)$ are solutions of (7.4.B) for which Parseval's equality holds and which have the same asymptotic form at infinity as $U(x, \lambda)$ $(\lambda \neq \lambda_0)$; for $\lambda = \lambda_0$,

$$U_2(x, \lambda) = e^{-|\lambda_0|x}[M_0+o(1)] \qquad (x \to \infty),$$

with $M_0 = T(\lambda_0)\hat{M}_0U_0^*$. Now $S(\lambda)$ also has the property I_s and according to §3, Chapter IV, the potential $V_2(x)$ of equation (7.4.B) may be determined by solving the fundamental integral equation

$$\mathbf{F}(x+y)+K_2(x, y)+\int_x^\infty K_2(x, t)\mathbf{F}(t+y)\,dt = 0, \qquad 0 < x \leq y,$$

where

$$\mathbf{F}(t) = M_0^2e^{-|\lambda_0|t}+F(t) = \sum_{k=0}^{p} M_k^2e^{-|\lambda_k|t}+F_s(t).$$

Since

$$\int_0^\infty x^{1+\theta} |V_2(x)|\, dx < \infty \qquad (|\theta| < \varepsilon),$$

it follows from the last equation and the results of § 2, Chapter III that both $\mathbf{F}_s'(x)$ and $F_s'(x)$ satisfy (7.6.1) (see (3.2.7) which is a consequence of the fundamental equation).

Now consider the homogeneous equation

$$u(t) + \int_0^\infty u(\xi)\mathbf{F}(t+\xi)\, d\xi = 0, \qquad 0 \leq t < \infty \tag{7.6.8}$$

and let us show that it has no non-trivial solution in $L^2_{(2)}(0, \infty)$. By the corollary to Lemma 3.4.1, the Fourier transform $\tilde{u}(\lambda)$ of any solution of this equation belonging to $L^2_{(2)}(0, \infty)$ must satisfy the relations

$$\tilde{u}(-\lambda)S(\lambda) = \tilde{u}(\lambda); \quad \tilde{u}(\lambda_k)M_k = 0 \qquad (k = 0, 1, \ldots, p). \tag{7.6.9}$$

Hence, according to (7.5.43) there results

$$\tilde{u}(-\lambda)T(-\lambda)\hat{S}(\lambda) = \tilde{u}(\lambda)T(\lambda); \quad \tilde{u}(\lambda_k)T(\lambda_k)M_k = 0 \qquad (k = 0, 1 \ldots, p)$$

Therefore $\tilde{u}(\lambda)\, T(\lambda)$ is the Fourier transform[7] of some solution of

$$v(t) + \int_0^\infty v(\xi)\hat{\mathbf{F}}(t+\xi)\, d\xi = 0, \qquad 0 \leq t < \infty.$$

whose kernel

$$\hat{\mathbf{F}}(t) = \sum_{k=0}^{p} \hat{M}_k^2 e^{-|\lambda_k|t} + \hat{F}_s(t)$$

is derived from the scattering data of the non-singular problem (7.5.C)—(0). But this last integral equation has no non-trivial solution. This implies that $\tilde{u}(\lambda)T(\lambda) \equiv 0$, hence $\tilde{u}(\lambda) \equiv 0$, and therefore $u(t) \equiv 0$.

We next show that $n_1 = 1$. By § 3, Chapter IV and the remark of Theorem 7.5.1, we can reconstruct the potential $V_1(x)$ of equation (7.3.A) in a unique manner by solving the integral equation

$$F(x+y) + K_1(x, y) + \int_x^\infty K_1(x, t)F(t+y)\, dt = 0, \qquad 0 < x \leq y.$$

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[7] From the expression (7.5.25), it follows that $\tilde{u}(\lambda)T(\lambda)$ is regular along with $\tilde{u}(\lambda)$ in the lower halfplane.

In this connection, if (7.6.5) had no non-trivial solutions, then by the remark of Theorem 5.4.1, $V_1(x)$ would satisfy the condition

$$\int_0 x\,|\,V_1(x)\,|\,dx<\infty,$$

which is impossible in the case under consideration ($a_{22} = 0$, $a_{12} \neq 0$). This implies that $n_1 \geq 1$. Suppose that (7.6.5) has at least two linearly independent solutions $x^{(1)}(t)$ and $x^{(2)}(t)$. By the corollary to Lemma 3.4.1, their Fourier transforms $\tilde{x}^{(i)}(\lambda)$ $(i = 1,2)$ must satisfy the relations

$$\tilde{x}^{(i)}(-\lambda)S(\lambda) = \tilde{x}^{(i)}(\lambda); \quad \tilde{x}^{(i)}(\lambda_k)M_k = 0 \qquad (7.6.10)$$
$$(k = 1,\ 2, \ldots, p).$$

We now form a non-trivial linear combination of these solutions $u(t) = \gamma_1 x^{(1)}(t) + \gamma_2 x^{(2)}(t)$ such that $\tilde{u}(\lambda_0)M_0 = 0$. This is possible since the rank of M_0 is one and therefore the equation

$$[\gamma_1\tilde{x}^{(1)}(\lambda_0) + \gamma_2\tilde{x}^{(2)}(\lambda_0)]M_0 = 0$$

has a non-trivial solution in γ_1, γ_2. The Fourier transform $\tilde{u}(\lambda)$ of $u(t)$ satisfies all of the relations in (7.6.9). But then $u(t)$ is a non-trivial solution of (7.6.8), which would contradict what was proved above. As a result, (7.6.5) cannot have two linearly independent solutions and $n_1 = 1$.

The Fourier transform $\tilde{x}(\lambda)$ of a solution of (7.6.5) satisfies the relations in (7.6.10) from which it follows that

$$\tilde{x}(-\lambda)T(-\lambda)\hat{S}(\lambda) = \tilde{x}(\lambda)T(\lambda); \quad \tilde{x}(\lambda_k)T(\lambda_k)M_k = 0$$
$$(k = 1,\ 2, \ldots, p).$$

Hence, applying the following factorization of the scattering matrix of the problem (7.5.C)—(0):

$$\hat{S}(\lambda) = E_3^{-1}(0,\ -\lambda)E_3(0,\ \lambda) = E_3^*(0,\ -\lambda)[E_3^*(0,\ \lambda)]^{-1} \qquad (7.6.11)$$

and reasoning as we did in the proof of Theorem 3.5.1, we conclude that

$$\tilde{x}(\lambda)T(\lambda) = \frac{2\lambda_0}{\lambda^2-\lambda_0^2}\,\tilde{x}(\lambda_0)T(\lambda_0)N_0E_3(0,\ \lambda)$$

and thus

$$\tilde{x}(\lambda) = \frac{2\lambda_0}{\lambda^2-\lambda_0^2}\,\tilde{x}(\lambda_0)T(\lambda_0)N_0E_3(0,\ \lambda)T(-\lambda). \qquad (7.6.12)$$

Here, $E_3(x, \lambda)$ is the special solution of (7.5.C), and N_0 the residue of $E_3^{-1}(0, \lambda)$ at the pole λ_0.

Taking into account that $E_3(0, \lambda) \to I$ and $T(-\lambda) \to I$ as $\lambda \to \infty$, we find from (7.6.12) that

$$\lim_{\lambda \to \infty} \lambda^2 \tilde{x}(\lambda)\,(I-P) = 2\lambda_0 \tilde{x}(\lambda_0) T(\lambda_0) N_0 (I-P).$$

By then considering the normalized matrix eigenfunction $Y_3(x, \lambda_0)$ of the boundary-value problem (7.5.C)—(0), we find as in the proof of Lemma 7.4.4 that $N_0 P = N_0$ and therefore

$$\lim_{\lambda \to \infty} \lambda^2 \tilde{x}(\lambda)\,(I-P) = 0.$$

Thus both statements concerning equation (7.6.5) have been proved.

Now consider equation (7.6.6). The Fourier transform $y(\lambda)$ of any solution of this equation is regular in the upper half-plane and is such that $\tilde{y}(-\lambda)S(\lambda) = -\tilde{y}(\lambda)$ for $-\infty < \lambda < \infty$ (Cf. (3.5.1)). According to (7.5.43) and (7.6.11), it follows from this that

$$\tilde{y}(-\lambda)T(-\lambda)E_3^*(0, -\bar{\lambda}) = -\tilde{y}(\lambda)T(\lambda)E_3^*(0, \bar{\lambda}) \qquad (\operatorname{Im} \lambda = 0).$$

The left-hand side of this equation is a regular function in the half-plane $\operatorname{Im} \lambda < 0$ except for the point $\lambda = -i\alpha$, whereas the right-hand side is regular for $\operatorname{Im} \lambda > 0$, $\lambda \neq i\alpha$. Therefore

$$\varphi(\lambda) = (\lambda^2+\alpha^2)\tilde{y}(\lambda)T(\lambda)E_3^*(0, \bar{\lambda})$$

is an entire function of λ such that

$$\varphi(-\lambda) = -\varphi(\lambda) \ (\operatorname{Im} \lambda = 0); \quad \varphi(\lambda) = o(\lambda^2) \qquad (\lambda \to \infty).$$

This implies that $\varphi(\lambda) = \lambda c$, where $c = (c_1, c_2)$ is a constant vector, and hence

$$\tilde{y}(\lambda) = \frac{\lambda}{\lambda^2+\alpha^2} c[E_3^*(0, \bar{\lambda})]^{-1} T(-\lambda).$$

Since $\tilde{y}(\lambda)$ is regular in the upper half-plane, we have (see (7.5.25))

$$c[E_3^*(0, -i\alpha)]^{-1} T(-i\alpha) = c[E_3^*(0, -i\alpha)]^{-1} (I-Q) = 0.$$

This implies that

$$c = \gamma q E_3^*(0, -i\alpha),$$

where q is the sole eigenvector of the matrix Q (a projection of rank one) and γ is a scalar factor. Hence the vector function

$$\tilde{y}(\lambda) = \frac{\lambda}{\lambda^2+\alpha^2}\gamma q E_3^*(0, \; -i\alpha)\,[E_3^*(0, \; \bar{\lambda})]^{-1}T(-\lambda)$$

is uniquely determined to within a scalar factor. This means that (7.6.6) either has only the trivial solution ($n_2 = 0$) or one non-trivial solution ($n_2 = 1$). If $n_2 = 1$, then at the poles $\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_p$ of $[E_3^*(0, \bar{\lambda})]^{-1}$ we must have

$$qE_3^*(0, \; -i\alpha)N_k^*T(-\bar{\lambda}_k) = 0 \qquad (k = 0, \; 1, \ldots, p),$$

where N_k is the residue of $E_3^{-1}(0, \lambda)$ at the point λ_k.

Noting that $T(-\lambda_k)(k = 0, \; 1, \; \ldots, \; p)$ is a non-singular matrix and that q is an eigenvector of Q, we find from this that

$$QE_3^*(0, \; -i\alpha)N_k^* = 0,$$

or

$$N_k E_3(0, \; -i\alpha)Q = 0 \qquad (k = 0, \; 1, \; 2, \ldots, p). \qquad (7.6.13)$$

Finally, let us consider the last equation (7.6.7). The Fourier transform $\tilde{z}(\lambda)$ of any solution of this equation must satisfy the relation $\tilde{z}(-\lambda)S(\lambda) = \tilde{z}(\lambda)$, from which it follows that $\tilde{z}(-\lambda)T(-\lambda)\,\hat{S}(\lambda) = \tilde{z}(\lambda)T(\lambda)$. This means that if $\tilde{z}(\lambda)$ is the Fourier transform of a solution of (7.6.7), then $\tilde{z}(\lambda)T(\lambda)$ is the Fourier transform of a solution of

$$w(t) + \int_0^\infty w(\xi)\hat{F}_s(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty. \qquad (7.6.14)$$

Since the number of linearly independent solutions of this equation is equal to the sum of the ranks of the matrices $\tilde{M}_k$ $(k = 0, 1, \ldots, p)$ [property V of the scattering data of the non-singular problem (7.5.C)-(0)], and since the rank of M_k is equal to that of $\tilde{\mathrm{M}}_k (k \neq 0)$ and the rank of $\tilde{M}_0$ is one, the number n_3 of linearly independent solutions of (7.6.7) does not exceed $1+\sum_{k=1}^{p} \operatorname{rank} M_k = r+1$.

On the other hand, if $\tilde{w}(\lambda)$ is the Fourier transform of a solution of (7.6.14), then $\tilde{z}(\lambda) = \tilde{w}(\lambda)T^{-1}(\lambda) = \tilde{w}(\lambda)T(-\lambda)$ is the Fourier transform of a solution of (7.6.7) if and only if $\tilde{w}(-i\alpha)Q = 0$. Suppose that $\tilde{w}^{(j)}(\lambda)$ $(j = 1, 2, \ldots, r+1)$ are the Fourier transforms of a fundamental system of linearly independent solutions of equation (7.6.14). In that case $\tilde{w}^{(j)}(-i\alpha) \neq 0$ for at least one value of j. Otherwise $\tilde{w}^{(j)}(\lambda)(\lambda^2+\alpha^2)^{-1}$ would also be Fourier transforms of solutions of (7.6.14), and hence the number of linearly independent solutions of this equation would exceed $r+1$, which is impossible. For definiteness, let us suppose that $\tilde{w}^{(1)}(-i\alpha) \neq 0$. Then for any $j = 2, \ldots, r+1$, the equation

$$[\gamma_j \tilde{w}^{(1)}(-i\alpha) - \tilde{w}^{(j)}(-i\alpha)]Q = 0$$

has a unique solution γ_j since Q is of rank one. Thus, the vector functions $[\gamma_j \tilde{w}^{(1)}(\lambda) - \tilde{w}^{(j)}(\lambda)]T(-\lambda)$ $(j = 2, \ldots, r+1)$ provide r Fourier transforms of linearly independent solutions of (7.6.7). This implies that $r \leq n_3 \leq r+1$.

Let us now establish the relationship between n_2 and n_3. Suppose that $n_2 = 0$. Then if $n_3 = r+1$, it would follow from the above discussion that the aggregate $S(\lambda)$, λ_k^2, and $M_k (k = 0, 1, \ldots, p)$ would possess all the properties of the scattering data of the non-singular boundary-value problem. This would mean that $U_2(0, \lambda) = 0$ which we know is actually not so. Therefore, $n_3 = r$ when $n_2 = 0$.

Now suppose that $n_2 = 1$. By the remark 2 of Theorem 3.5.1, the Fourier transforms $\tilde{w}^{(j)}(\lambda)$ $(j = 1, 2, \ldots, r+1)$ of linearly independent solutions of (7.6.14) are expressible in terms of $E_3(0, \lambda)$ by means of the relations

$$\tilde{w}^{(j)}(\lambda) = \sum_{k=0}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{w}^{(j)}(\lambda_k) N_k E_3(0, \lambda) \qquad (j = 1, 2, \ldots, r+1).$$

On the other hand, when $n_2 = 1$, (7.6.13) holds from which it folows that

$$\tilde{w}^{(j)}(-i\alpha)Q = \sum_{k=0}^{p} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{w}^{(j)}(\lambda_k) N_k E_3(0, -i\alpha)Q = 0$$
$$(j = 1, \ldots, r+1).$$

Therefore the vector functions $\tilde{z}^{(j)}(\lambda) = \tilde{w}^{(j)}(\lambda)T(-\lambda)$ $(j = 1, 2, \ldots, r+1)$, are the Fourier transforms of solutions of (7.6.7) which are clearly linearly independent. This means that $n_3 = r+1$ when $n_2 = 1$ and the proof of the lemma is complete.

LEMMA 7.6.2: *Let* $\tilde{x}(\lambda)$ *be the Fourier transform of the non-trival solution of* (7.6.5). *If* $S(0) = I$, *then* $\tilde{x}(0)(I-P) \neq 0$.

Proof: By (7.6.12), we have

$$\tilde{x}(\lambda) = (\lambda^2 - \lambda_0^2)^{-1} c N_0 E_3(0, \lambda) T(-\lambda),$$

where c is a constant vector such that $cN_0 \neq 0$. Since $N_0P = N_0$, $cN_0 = cN_0P = dP$, where d is a non-zero vector. Therefore

$$\tilde{x}(0) = -\lambda_0^{-2} dPE_3(0)T(0) \qquad (E_3(0) = E_3(0, 0)),$$

and inasmuch as $dP \neq 0$, to prove the lemma it is sufficient to show that

$$P_3E(0)T(0)(I-P) \neq 0. \tag{7.6.15}$$

To this end, we first of all note that by Lemma 7.5.1, the matrix $Y_3(x)$ obtained from $Y_2(x)$ by means of the transformation (7.5.3) vanishes at $x = 0$. Since $Y_3(x)$ is a solution of (7.5.C) for $\lambda = 0$,

$$Y_3(x) = G_3(x)A, \tag{7.6.16}$$

where $G_3(x)$ is the solution of (7.5.C) for $\lambda = 0$ satisfying the conditions $G_3(0) = 0$ and $G_3'(0) = I$, and A is a non-zero constant matrix.

The matrix A satisfies the following relations:

$$AP = A; \quad (I-P)A \neq 0. \tag{7.6.17}$$

For, since $Y_1(x)P \equiv Y_1(x)$, it follows from the form of the transformations (7.4.6) and (7.5.3) that $Y_3(x) \equiv Y_3(x)P$, and this implies that $AP = A$. Furthermore,

$$\lim_{x\to 0} W\{Y_3^*(x, -i\alpha), Y_3(x)\} = Y_3^*(0, -i\alpha)Y_3'(0) = Y_3^*(0, -i\alpha)A.$$

Now, suppose $(I-P)A = 0$. Because $Y_3^*(0, -i\alpha) = \alpha^2(I-P)$ (see (7.5.33)), we would have

$$\lim_{x\to 0} W\{Y_3^*(x, -i\alpha), Y_3(x)\} = \alpha^2(I-P)A = 0.$$

But then the transformation (7.5.14) applied to $Y_3(x)$ would be expressible as

$$Y_2(x) = Y_3(x) + Y_2(x, -i\alpha)\int_0^x Y_3^*(t, -i\alpha)Y_3(t)\,dt,$$

which would mean that $Y_2(0) = 0$. And this is not true. Therefore, $(I-P)A \neq 0$.

Now $Y_1(x) = -3xP + o(x^{1-\theta})P$ as $x \to \infty$, and the transformation (7.4.6) preserves asymptotic behavior at infinity. Hence, repeating computations carried out in the proof of Lemma 7.5.4, we obtain

$$Y_3(x) = T(0)[-3xP + o(x^{1-\theta})P] \qquad (x \to \infty).$$

On the other hand, Lemma 1.5.1 and formula (7.6.16) yield

$$Y_3(x) = xE_3^*(0)A + o(x^{1-\theta}) \qquad (x \to \infty).$$

As a consequence,

$$E_3^*(0)A = -3T(0)P.$$

But since det $E_3^*(0) \neq 0$ when $S(0) = \hat{S}(0) = I$ (see the corollary to Theorem 5.6.2), we then have that

$$A = -3[E_3^*(0)]^{-1}T(0)P$$

and hence by (7.6.17)

$$(I-P)[E_3^*(0)]^{-1}T(0)P \neq 0.$$

But $T(0) = T^*(0) = T^{-1}(0) \neq 0$, and this last inequality yields

$$P[E_3(0)T(0)]^{-1}(I-P) \neq 0,$$

from which follows (7.6.15), and the lemma is proved.

§ 7. Behavior of the Scattering Matrix When $\lambda \to 0$. Summary of Results

1. To conclude our investigation of the problem (7.1.0)—(0) carried out in the preceding sections, we shall now study the behavior of its scattering matrix at $\lambda = 0$.

LEMMA 7.7.1: *The scattering matrix* $S(\lambda)$ *of the boundary-value problem* (7.1.0)—(0) *has a limit* $S(0)$ *as* $\lambda \to 0$, *and this limit is either the identity matrix or the matrix* $2P-I$*:*

$$S(0) = \lim_{\lambda \to 0} S(\lambda) = \pm(I-P)+P. \tag{7.7.1}$$

Proof: Keeping the notations of §§ 3—5 of this chapter, we consider the solution $U_1(x, \lambda)$ $(\lambda > 0)$ of equation (7.3.A) having the asymptotic behavior

$$U_1(x, \lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (x \to \infty).$$

From this and Theorem 1.4.1, it follows that

$$U_1(x, \lambda) = E_1(x, -\lambda) - E_1(x, \lambda)S(-\lambda) \qquad (\lambda > 0), \tag{7.7.2}$$

where

$$E_1(x, \lambda) = e^{-i\lambda x}I + \int_x^\infty K_1(x, t)e^{-i\lambda t}\,dt \qquad (\operatorname{Im}\lambda \leq 0) \tag{7.7.3}$$

is the special solution of equation (7.3.A).

By the use of (7.7.3) and the estimate for $K_1(x, t)$ corresponding to (1.3.3), it is easy to show that

$$\lim_{\lambda \to 0} E_1(x, \lambda) = E_1(x, 0) = I + \int_x^\infty K_1(x, t)\, dt,$$

uniformly in each segment $0 < x_1 \leq x \leq x_2$. Furthermore, from its properties proved at the beginning of the preceding section, it follows that $S(\lambda)$ is continuous over the entire real axis. Therefore, if λ_n is any sequence of positive numbers tending to zero, upon taking $\lambda = \lambda_n$ in (7.7.2) and letting $n \to \infty$, we obtain the matrix

$$U_1(x) = \lim_{n \to \infty} U_1(x, \lambda_n) = E_1(x, 0)[I - S(0)], \tag{7.7.4}$$

defined for all $x > 0$. Moreover, the sequence $U_1(x, \lambda_n)$ converges to $U_1(x)$ uniformly in each interval $0 < x_1 \leq x \leq x_2$.

Let us now consider the solution $U(x, \lambda)$ of equation (7.1.0) ($\lambda > 0$) obtained from $U_1(x, \lambda)$ by means of the transformation (7.3.17), assuming henceforth that we are in the most involved case $a_{22} = 0$, $a_{12} \neq 0$. By (7.5.35) and the remark of Theorem 7.5.1, in this case

$$\begin{aligned} U(x, \lambda) = U_1(x, \lambda) + Y(x) \int_0^x Y_1^*(t) U_1'(t, \lambda)\, dt + \\ + |a_{12}|^{-2} m^{-2} Y(x) Y_1^*(0) U_1(0, \lambda) \end{aligned} \tag{7.7.5}$$

(if $a_{22} \neq 0$, or if $a_{22} = a_{12} = 0$, then $U_1(0, \lambda) = 0$ and $U(x, \lambda)$ has the correspondingly simpler form (7.3.19)).

Since $U_1(x, \lambda) = \hat{U}_1(x, \lambda)\, T(\lambda)$, where $T(\lambda)$ is a unitary matrix, (7.5.39) shows that $U_1(x, \lambda)$ is uniformly bounded in the region $0 \leq x < \infty$, $0 < \lambda < \infty$. From this and (7.7.4), it follows that

$$\lim_{n \to \infty} \int_0^x Y_1^*(t) U_1(t, \lambda_n)\, dt = \int_0^x Y_1^*(t) U_1(t)\, dt$$

uniformly in each interval $0 < x_1 \leq x \leq x_2$. Because of the boundedness of $U_1(0, \lambda)$ for $0 < \lambda < \infty$, we may assume the existence of

$$\lim_{n \to \infty} U_1(0, \lambda_n) = U_1(0)$$

without any loss of generality.

From (7.7.5), it follows that $U(x, \lambda_n)$ converges to the matrix

$$U(x) = \lim_{n\to\infty} U(x, \lambda_n) = U_1(x) + \tag{7.7.6}$$
$$+Y(x) \int_0^x Y_1^*(t) U_1(t)\, dt + |a_{12}|^{-2} m^{-2} Y(x) Y_1^*(0) U_1(0)$$

uniformly in each finite interval $0 < x_1 \leq x \leq x_2$.

Using the equation

$$U''(x, \lambda_n) - [V(x) + 6x^{-2}P] U(x, \lambda_n) + \lambda_n^2 U(x, \lambda_n) = 0,$$

we find that the sequence $U''(x, \lambda_n)$ is also uniformly convergent in each finite interval $0 < x_1 \leq x \leq x_2$, and hence $U''(x)$ exists for all positive values of x and

$$U''(x) - [V(x) + 6x^{-2}P] U(x) = 0.$$

Thus, the matrix $U(x)$ determined by (7.7.6) is a solution of (7.1.0) for $\lambda = 0$. Now from (7.7.6) in conjunction with (7.3.2), (7.3.6), and (7.7.4), we find that as $x \to \infty$

$$U(x) = I - S(0) + o(1) + Y(x) \left\{ \text{const} + \int_N^x Y_1^*(t) [I - S(0) + o(1)]\, dt \right\} =$$
$$= \left(I - \frac{3}{2} P\right) [I - S(0)] + o(1), \tag{7.7.7}$$

since for N sufficiently large and $x \to \infty$

$$Y(x) \int_N^x Y_1^*(t)\, dt =$$
$$= [x^{-2} I + o(x^{-2-\theta})] P \int_N^x [-3tI + o(t^{1-\theta})]\, dt = -\frac{3}{2} P + o(1).$$

Thus, $U(x)$ is a solution of (7.1.0) for $\lambda = 0$ bounded at infinity. But then, according to Theorem 7.2.1,

$$\lim_{x\to\infty} PU(x) = 0.$$

This and (7.7.7) show that

$$P\left(I - \frac{3}{2} P\right) [I - S(0)] = -\frac{1}{2} P [I - S(0)] = 0$$

or

$$PS(0)=P. \tag{7.7.8}$$

Now consider the relations

$$S(-\lambda) = S^*(\lambda), \qquad S(-\lambda)S(\lambda) = I.$$

Letting λ approach zero, we obtain $S(0) = S^*(0)$ and $[S(0)]^2 = I$, and this in conjunction with (7.7.8) leads us to the result (7.7.1).

REMARK. If $\lim\limits_{\lambda\to 0} S(\lambda) \neq I$, then the problem (7.1.0)—(0) has a virtual level, i.e., a non-trivial bounded vector solution of the system (7.1.0) exists for $\lambda = 0$ which vanishes at $x = 0$. But it is not in $L^2_{(2)}(0, \infty)$. However, if $\lim\limits_{\lambda\to 0} S(\lambda) = I$, then the problem (7.1.0)—(0) will have no virtual level.

We shall not prove these statements. We merely observe that they may be shown to hold by considering the non-singular boundary-value problem to which the problem (7.1.0)—(0) can be reduced by the transformations of § § 3—5.

2. We have completed our analysis of the problem (7.1.0)—(0), and in conclusion we present a final summary of the results obtained.

By Theorems 7.3.1, 7.4.1, and 7.5.1, the boundary-value problem (7.1.0)—(0) has a continuous spectrum over the entire positive axis ($\lambda^2 > 0$) and, possibly, a finite number of non-positive eigenvalues $\lambda_1^2 < \lambda_2^2 < \ldots < \lambda_p^2 \leq 0$. A complete set of normalized matrix eigenfunctions exists for the problem satisfying the Parseval equality

$$\int_0^\infty \Phi(x)\Phi^*(x)dx = \sum_{k=1}^{p} U(\Phi;\ \lambda_k)U^*(\Phi;\ \lambda_k) +$$

$$+\frac{1}{2\pi}\int_0^\infty U(\Phi;\ \lambda)U^*(\Phi;\ \lambda)\, d\lambda,$$

$$U(\Phi;\ \lambda) = \text{l.i.m.} \int_0^\infty \Phi(x)U(x,\ \lambda)\, dx.$$

As $x \to \infty$, these solutions have the asymptotic behavior

$$U(x,\ \lambda) = e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0),$$

$$U(x,\ \lambda_k) = e^{-|\lambda_k|x}[M_k + o(1)] \qquad (\lambda_k^2 < 0)$$

and if $\lambda_p = 0$,

$$U(x,\ 0) = x^{-2}[M_p + o(1)] \qquad (M_p = mP,\ m > 0).$$

The unitary matrix $S(\lambda)$ is the scattering matrix, the non-negative Hermitian matrices M_k the normalization matrices, and the aggregate of quantities $S(\lambda)$, λ_k^2, M_k $(k = 1, 2, \ldots, p)$ the scattering data of the problem (7.1.0)—(0).

$S(\lambda)$ has the following two properties (see the beginning of § 6 of Chapter VII and Lemma 7.7.1):

1°. The matrix $I-S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_s(t)$, such that

$$F_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [I-S(\lambda)]e^{i\lambda t}\, d\lambda$$

is summable over the entire real axis, and for all positive values of t, $F_s'(t)$ exists and

$$\int_0^{\infty} t^{1+\theta} \,|\, F_s'(t) \,|\, dt < \infty \qquad (-\varepsilon < \theta < \varepsilon).$$

2°. $S(0) = \pm\, (I-P)+P$.

The further properties of the scattering data are associated with the equations

$$\begin{aligned} x(t)+\int_0^{\infty} x(\xi)F(t+\xi)\, d\xi &= 0, \qquad 0 \leq t < \infty, \qquad (7.7.9)\\ -y(t)+\int_{-\infty}^{0} y(\xi)F_s(t+\xi)\, d\xi &= 0, \qquad -\infty < t \leq 0,\\ z(t)+\int_0^{\infty} z(\xi)F_s(t+\xi)\, d\xi &= 0, \qquad 0 \leq t < \infty, \end{aligned}$$

where

$$F(t) = \sum_k{}' M_k^2 e^{-|\lambda_k|t}+F_s(t)$$

and the summation extends over all non-zero eigenvalues of the problem in question.

Let n_1, n_2, and n_3 be the respective number of linearly independent vector solutions of each of these equations, $\tilde{x}(\lambda)$ the Fourier transform of any non-trivial solution of (7.7.9), r the sum of the ranks of the normalization matrices, $M_1, M_2, \ldots, M_p$, and $K(x, y)$ the matrix solution of

$$F(x+y)+K(x, y)+\int_x^\infty K(x, t)F(t+y)\,dt = 0 \qquad (0 < x \leq y)$$

(according to Theorem 3.4.1, a unique solution exists for any $x > 0$).

The following three cases are then possible (see the corollary to and the remark of Theorem 7.3.1, the corollary to Theorem 7.4.1, and Lemmas 7.4.4, 7.6.1, and 7.6.2):

(a) all $\lambda_k \neq 0$, $n_1 = 0$, $n_2 = 0$, $n_3 = r$, and

$$P\left[I+\int_0^\infty K(0, t)\,dt\right] = \gamma P, \tag{7.7.10}$$

where γ is a non-zero number;

(b) $\lambda_p = 0$, $M_p = mP$ $(m > 0)$, $n_1 = 1$, $n_2 = 0$, $n_3 = r$ and

$$\lim_{\lambda\to\infty} \lambda^2\tilde{x}(\lambda)\,(I-P) = 0;$$

(c) all $\lambda_k \neq 0$, $n_1 = 1$, $n_2 \leq 1$, $n_3 = n_2+r$, $\lim\limits_{\lambda\to\infty} \lambda^2\,\tilde{x}(\lambda)(I-P) \neq 0$ and if $S(0) = I$, then $\tilde{x}(0)(I-P) = 0$. As to which of the three cases occurs depends on the behavior of $Y(x)$ in the neighborhood of the origin, $Y(x)$ being the non-trivial solution of (7.1.0) ($\lambda = 0$) which vanishes at infinity: if $Y(x)$ is unbounded when $x \to 0$, then case (a) holds, if $Y(0) = 0$, then case (b) holds, and if $Y(0)$ is bounded but $Y(0) \neq 0$, then the third case occurs.

Furthermore, in case (c) there are two possibilities related to the behavior of the matrix eigenfunctions $U(x, \lambda_k)$ at $x = 0$: if for at least one value of $k = 1, 2, \ldots, p$

$$U'(0, \lambda_k) \neq 0,$$

then $n_2 = 0$; however, if

$$U'(0, \lambda_k) = 0 \qquad (k = 1, 2, \ldots, p),$$

then $n_2 = 1$. We leave the proof of this fact to the reader.

CHAPTER VIII

Reconstruction of the Singular Boundary-Value Problem from Its Scattering Data

§ 1. Case (a)

1. In the preceding chapter it was shown that the scattering data of the problem (7.1.0)—(0) satisfies the conditions 1°, 2°, and one of the conditions (a), (b), or (c) (see Chapter VII, § 7, sub-section *2*). It will now be shown that these conditions are not only necessary but also sufficient for a unitary matrix $S(\lambda)$, Hermitian matrices $M_1, M_2, \ldots, M_p$ and numbers $\lambda_1^2 < \lambda_2^2 < \ldots < \lambda_p^2 \leq 0$ to be the scattering data for some boundary-value problem (7.1.0)—(0). At the same time, an algorithm will be given for reconstructing the problem from its scattering data. It is clear that each of the three possible cases will have to be treated individually.

Consider any unitary matrix $S(\lambda)$, negative numbers λ_k^2, and Hermitian matrices M_k $(k = 1, 2, \ldots, p)$ with the properties 1°, 2°, and (a). According to Chapter V, the quantities

$$S(\lambda),\ \lambda_k^2,\ M_k \qquad (k=1,\ 2,\ldots,\ p) \tag{8.1.1}$$

are then the scattering data for some uniquely determined non-singular boundary-value problem with a Hermitian potential $V_1(x)$ satisfying the condition[1]

$$\int_0^\infty x^{1+\theta}\,|\,V_1(x)\,|\,dx<\infty \qquad (-\varepsilon<\theta<\varepsilon). \tag{8.1.2}$$

---

[1] The condition (8.1.2) for $V_1(x)$ follows from the analogous one for $F'(t)$, where

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k|t} + \frac{1}{2\pi}\int_{-\infty}^{\infty} [I - S(\lambda)] e^{i\lambda t}\, d\lambda$$

(see the property 1°), and from the inequality (5.4.4′) in the remark of Theorem 5.4.1.

Let $E_1(x, \lambda)$ denote the special solution of

$$Y'' - V_1(x)Y + \lambda^2 Y = 0, \tag{8.1.3}$$

so that (see Theorem 1.3.1 and § 1, Chapter III)

$$E_1(x, \lambda) = e^{-i\lambda x}I + \int_x^\infty K_1(x, t)e^{-i\lambda t}\,dt \qquad (\operatorname{Im}\lambda \leq 0),$$

with $K_1(x, y)$ the solution of the equation

$$F(x+y) + K_1(x, y) + \int_x^\infty K_1(x, t)F(t+y)\,dt = 0,$$

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k|t} + \frac{1}{2\pi}\int_{-\infty}^{\infty} [I - S(\lambda)]e^{i\lambda t}\,d\lambda,$$

and set $E_1(0) = E_1(0, 0)$. According to property (a), we have (see (7.7.10))

$$PE_1(0) = P\left[I + \int_0^\infty K_1(0, t)\,dt\right] = \gamma P, \tag{8.1.4}$$

where γ is a non-zero number.

Suppose further that $G_1(x)$ is the solution of (8.1.3) for $\lambda = 0$ satisfying the conditions $G_1(0) = 0$ and $G_1'(0) = I$, so that as $x \to 0$ (see Theorem 1.2.1)

$$G_1(x) = x[I + o(x^\theta)], \quad G_1'(x) = I + o(x^\theta) \qquad (\theta < \varepsilon). \tag{8.1.5}$$

According to Lemma 1.5.1,

$$G_1(x) = xE_1^*(0) + o(x^{1-\theta}), \quad G_1'(x) = E_1^*(0) + o(x^{-\theta}) \qquad (\theta < \varepsilon)$$

as $x \to \infty$, and from this and (8.1.4), it follows that

$$G_1(x)P = \bar{\gamma}xP + o(x^{1-\theta})P \qquad (\theta < \varepsilon) \tag{8.1.6}$$

as $x \to \infty$; moreover, this asymptotic relation may be differentiated.

2. Define

$$Y_1(x) = -3\bar{\gamma}^{-1}G_1(x)P, \tag{8.1.7}$$

$$Y(x) = Y_1(x)\left[I - P - \int_0^x Y_1^*(t)Y_1(t)\,dt\right]^{-1}, \tag{8.1.8}$$

and consider the transformation

$$\Phi(x, \lambda) = \Phi_1(x, \lambda) - \lambda^{-2}Y(x)W\{Y_1^*(x), \Phi_1(x, \lambda)\} \tag{8.1.9}$$

(see (7.3.17)). According to Theorem 6.2.1, it takes any solution $\Phi_1(x, \lambda)$ of (8.1.3) into a solution of

$$Y'' - \{V_1(x) + 2[Y(x)Y_1^*(x)]'\}Y + \lambda^2 Y = 0. \tag{8.1.10}$$

Now, (8.1.5)—(8.1.7) show that for any $\theta < \varepsilon$

$$\left.\begin{aligned} Y_1(x) &= [-3\bar{\gamma}^{-1}xI + o(x^{1+\theta})]P, \\ Y_1'(x) &= [-3\bar{\gamma}^{-1}I + o(x^{\theta})]P \end{aligned}\right\} \quad (x \to 0), \tag{8.1.11}$$

$$Y_1(x) = [-3xI + o(x^{1-\theta})]P, \; Y_1'(x) = [-3I + o(x^{-\theta})]P \quad (x \to \infty),$$

and it is easily verified from (8.1.8) that

$$\left.\begin{aligned} Y(x) &= [\gamma x^{-2}I + o(x^{-2+\theta})]P, \\ Y'(x) &= [-2\gamma x^{-3}I + o(x^{-3+\theta})]P \end{aligned}\right\} \quad (x \to 0), \tag{8.1.12}$$

$$Y(x) = [x^{-2}I + o(x^{-2-\theta})]P, \; Y'(x) = [-2x^{-3}I + o(x^{-3-\theta})]P \quad (x \to \infty).$$

Hence,

$$\begin{aligned} 2[Y(x)Y_1^*(x)]' &= 6x^{-2}P + o(x^{-2+\theta}) \qquad (x \to 0), \\ 2[Y(x)Y_1^*(x)]' &= 6x^{-2}P + o(x^{-2-\theta}) \qquad (x \to \infty). \end{aligned} \tag{8.1.13}$$

Writing equation (8.1.10) in the form

$$Y'' - [V(x) + 6x^{-2}P]Y + \lambda^2 Y = 0, \tag{7.1.0}$$

in which

$$V(x) = V_1(x) + 2[Y(x)Y_1^*(x)]' - 6x^{-2}P, \tag{8.1.14}$$

we conclude that $V(x)$ is Hermitian along with $V_1(x)$ and satisfies (7.1.1.) by virtue of (8.1.2) and (8.1.13).

We now show that, the boundary condition $Y(0, \lambda) = 0$ is preserved under the transformation (8.1.9). If $\Phi_1^{(0)}(x, \lambda)$ is a solution of (8.1.3) such that $\Phi_1^{(0)}(0, \lambda) = 0$, then by (8.1.11) we obviously have

$$\lim_{x \to 0} W\{Y_1^*(x), \Phi_1^{(0)}(x, \lambda)\} = 0.$$

Therefore, according to Lemma 6.3.1, the transformation (8.1.9) applied to $\Phi_1^{(0)}(x, \lambda)$ is expressible as

$$\Phi^{(0)}(x, \lambda) = \Phi_1^{(0)}(x, \lambda) + Y(x) \int_0^x Y_1^*(t)\Phi_1^{(0)}(t, \lambda)\, dt. \tag{8.1.15}$$

Making use of (8.1.11), (8.1.12), and Theorem 1.2.1, we conclude from this that $\Phi^{(0)}(0, \lambda) = 0$.

It is also easy to see that the transformation (8.1.9) does not effect the asymptotic form of solutions at infinity (see Lemma (7.3.4)).

Now formula (8.1.15) can be applied, in particular, to the solutions $U_1(x, \lambda)$ of equation (8.1.3) ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1, 2, \ldots, p$), for which the following Parseval equality holds:

$$\sum_{k=1}^{p} U_1(\Phi;\ \lambda_k)U_1^*(\Phi;\ \lambda_k) + \\ +\frac{1}{2\pi}\int_0^\infty U_1(\Phi;\ \lambda)U_1^*(\Phi;\ \lambda)\,d\lambda = \int_0^\infty \Phi(x)\Phi^*(x)\,dx, \qquad (8.1.16)$$

$$U_1(\Phi;\ \lambda) = \text{l.i.m.}\int_0^\infty \Phi(x)U_1(x,\ \lambda)\,dx$$

and which behave at infinity like

$$\begin{aligned} U_1(x,\ \lambda) &= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0), \\ U_1(x, \lambda_k) &= e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 1,\ 2, \ldots, p). \end{aligned} \qquad (8.1.17)$$

There results

$$U(x,\ \lambda) = U_1(x,\ \lambda) + Y(x)\int_0^x Y_1^*(t)U_1(t,\ \lambda)\,dt$$

$$(\lambda > 0;\quad \lambda = \lambda_k,\quad k = 1,\ 2, \ldots, p).$$

These matrices satisfy equation (8.1.10), the boundary condition $U(0, \lambda) = 0$, and at infinity are asymptotically equal to the functions in (8.1.17). Applying Theorem 6.3.1, we further find that a Parseval equality analagous to (8.1.16) holds for these matrices.

This means that if the data given in (8.1.1) has the properties 1°, 2°, and (a), then it comprises the scattering data for the problem (7.1.0)—(0) with a potential $V(x)$ which can be determined from (8.1.14).

§ 2. Case (b)

1. Consider any given unitary matrix $S(\lambda)$, negative numbers $\lambda_1^2, \lambda_2^2, \ldots, \lambda_{p-1}^2$ and Hermitian matrices $M_1, M_2, \ldots, M_p$ and let $\lambda_p^2 = 0$. Suppose now that the conditions 1°, 2°, and (b) hold.

By the condition (b), the equation

$$x(t)+\int_0^\infty x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty, \tag{8.2.1}$$

in which

$$F(t) = \sum_{k=1}^{p-1} M_k^2 e^{-|\lambda_k|t}+\frac{1}{2\pi}\int_{-\infty}^{\infty}[I-S(\lambda)]e^{i\lambda t}\,d\lambda,$$

has a non-trivial vector solution $x(t)$ which is unique to within a scalar factor; moreover, its Fourier transform $\tilde{x}(\lambda)$ is such that

$$\lim_{\lambda\to\infty} \lambda^2\tilde{x}(\lambda)\,(I-P) = 0. \tag{8.2.2}$$

It follows from this that $\tilde{x}(\lambda)P\neq 0$, otherwise $\lambda^2\tilde{x}(\lambda)$ would be the Fourier transform of some solution of (8.2.1).[2] Therefore, there exists a negative quantity $\lambda_0^2(\operatorname{Im}\lambda_0 < 0)$ distinct from all λ_k^2 such that

$$\tilde{x}(\lambda_0)P\neq 0.$$

Letting

$$M_0=P,$$
$$\mathbf{F}(t) = F(t)+M_0^2 e^{-|\lambda_0|t},$$

we find that the equation

$$x(t)+\int_0^\infty x(\xi)\mathbf{F}(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty$$

has no non-trivial solution because of the corollary to Lemma 3.4.1.

Now taking into consideration that the given data has the properties 1° and (b) and that M_0 and M_p are of the same rank, we

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[2] According to the corollary to Lemma 3.4.1, the vector $x(t)$ is a solution of (8.2.1) if the following conditions hold:

$$\tilde{x}(-\lambda)S(\lambda) = \tilde{x}(\lambda) \quad (\operatorname{Im}\lambda = 0), \quad \tilde{x}(\lambda_k)M_k = 0 \quad (k = 1,2,\ldots,p-1).$$

It is clear that both $\tilde{x}(\lambda)$ and $\lambda^2\tilde{x}(\lambda)$ satisfy these conditions, and also that both these functions are regular in the lower halfplane.

Finally, by the remark 2 of Theorem 3.5.1, $\tilde{x}(\lambda)$ can be represented in the form (3.5.6) from which it follows that $\lambda^2\tilde{x}(\lambda)$ may tend to zero no slower than λ^{-2} when $\lambda\to\infty$. Thus, if $\lim_{\lambda\to\infty}\lambda^2\tilde{x}(\lambda) = 0$, then $\lambda^2\tilde{x}(\lambda)$ would be the Fourier transform of some solution of (8.2.1).

conclude that $S(\lambda)$, λ_k^2, and M_k ($k = 0, 1, \ldots, p-1$) comprise the scattering data of a non-singular boundary-value problem. Moreover, the Hermitian potential $V_2(x)$ of this problem satisfies the condition

$$\int_0^\infty x^{1+\theta} \mid V_2(x) \mid dx < \infty \qquad (-\varepsilon < \theta < \varepsilon) \tag{8.2.3}$$

(see § 1 of Chapter V and footnote 1 of this chapter).

Let $E_2(x, \lambda)$ be the special solution of

$$Y'' - V_2(x)Y + \lambda^2 Y = 0 \tag{8.2.4}$$

and set $E_2(\lambda) = E_2(0, \lambda)$. By the same reasoning used in the proof of Theorem 3.5.1, it can be shown that (Cf. (7.4.27))

$$\tilde{x}(\lambda) = \frac{2\lambda_0}{\lambda^2 - \lambda_0^2} \tilde{x}(\lambda_0) N_0 E_2(\lambda) = \frac{1}{\lambda^2 - \lambda_0^2} c N_0 E_2(\lambda),$$

where c is any constant vector, and N_0 is the residue at the simple pole λ_0 of $E_2^{-1}(\lambda)$. This and (8.2.2) imply that $N_0(I-P) = 0$ or $N_0 P = N_0$. Furthermore, $E_2(\lambda_0)P = E_2(\lambda_0)M_0 = 0$, i.e., P is a projection onto the null space of the matrix $E_2(\lambda_0)$. Hence, by (3.1.7), $PN_0 = N_0$ and so

$$N_0 = PN_0P = \nu_0 P, \tag{8.2.5}$$

where ν_0 is some non-zero number.

2. Consider the matrix

$$Y_2(x, \lambda_0) = E_2(x, \lambda_0) M_0 = E_2(x, \lambda_0) P \tag{8.2.6}$$

consisting of one non-zero column, a normalized vector eigenfunction of the problem (8.2.4)—(0) corresponding to the eigenvalue λ_0^2. Since by (3.1.8)

$$E_2'(0, \lambda_0)P = 2\lambda_0 N_0^* A_0 P,$$

where

$$A_0 = \int_0^\infty E_2^*(x, \lambda_0) E_2(x, \lambda_0)\, dx$$

(P is a projection onto the null space of the matrix $E_2(\lambda_0)$), it follows that

$$Y_2'(0, \lambda_0) = E_2'(0, \lambda_0)P = 2\lambda_0 \bar{\nu}_0 P A_0 P = \nu P$$

(see (8.2.5)). Here, $\nu \neq 0$ because A_0 is a positive definite matrix.

From this last result and (8.2.6), we conclude that

$$Y_2(x, \lambda_0) = \nu G_2(x, \lambda_0)P,$$

in which $G_2(x, \lambda_0)$ is the solution of (8.2.4) for $\lambda = \lambda_0$ satisfying the conditions $G_2(0, \lambda_0) = 0$ and $G_2'(0, \lambda_0) = I$. Therefore, for any $\theta < \varepsilon$, we have in virtue of (8.2.3) and Theorem 1.2.1

$$Y_2(x, \lambda_0) = [\nu x I + o(x^{1+\theta})]P, \; Y_2'(x, \lambda_0) = [\nu I + o(x^{\theta})]P \quad (x \to 0). \tag{8.2.7}$$

Now let $G_2(x)$ be the solution of (8.2.4) for $\lambda = 0$ satisfying the conditions $G_2(0) = 0$ and $G_2'(0) = I$. By Lemma 1.5.1, this solution has the asymptotic behavior,

$$G_2(x) = xE_2^*(0) + o(x^{1-\theta}), \; G_2'(x) = E_2^*(0) + o(x^{-\theta}) \quad (x \to \infty), \tag{8.2.8}$$

for any $\theta < \varepsilon$.

LEMMA 8.2.1: *If* det $E_2(0) = 0$, *then*

$$E_2(0) = E_2(0)P \neq 0.$$

Proof: If det $E_2(0) = 0$, then by the corollary to Theorem 5.6.2, $S(0) \neq I$ and by 2°

$$S(0) = -(I-P)+P. \tag{8.2.9}$$

This implies that the rank of $I-S(0)$ is one, and since the dimension of the null space of the matrix $E_2(0)$ is equal to the rank of $I-S(0)$ (see (5.6.20)), it follows that $E_2(0) \neq 0$.

Now consider the relation

$$E_2(\lambda) = E_2(-\lambda)S(\lambda),$$

which is a consequence of the factorization of the matrix $S(\lambda)$ (Cf. (2.3.3)). Letting λ approach zero in this and taking into consideration (8.2.9), we obtain

$$E_2(0) = E_2(0)[-(I-P)+P].$$

Thus, $E_2(0)(I-P) = 0$, and the lemma is proved.

LEMMA 8.2.2: *There exists a matrix A such that*

$$E_2^*(0)A = -3P, \qquad AP = A.$$

If det $E_2(0) \neq 0$, *then these relations determine A uniquely; however, if* det $E_2(0) = 0$, *then there exists a one-parameter family of matrices satisfying these relations.*

Proof: When det $E_2(0) \neq 0$, we evidently can take

$$A = -3[E_2^*(0)]^{-1}P.$$

If $\det E_2(0) = 0$, then according to Lemma 8.2.1,

$$E_2(0) = \begin{pmatrix} 0 & e_{12} \\ 0 & e_{22} \end{pmatrix}, \quad |e_{12}|^2 + |e_{22}|^2 \neq 0$$

and it is easy to see that the matrix

$$A = \begin{cases} \begin{pmatrix} 0 & -(3+\beta\bar{e}_{22})\bar{e}_{12}^{-1} \\ 0 & \beta \end{pmatrix}, & \text{if} \quad e_{12} \neq 0, \\ \begin{pmatrix} 0 & \beta \\ 0 & -3\bar{e}_{22}^{-1} \end{pmatrix}, & \text{if} \quad e_{12} = 0, \end{cases}$$

where β is an arbitrary parameter, satisfies the necessary requirements.

Let A be any matrix for which the conditions of Lemma 8.2.2 hold, and set

$$Y_2(x) = G_2(x)A. \tag{8.2.10}$$

Then for any $\theta < \varepsilon$, we have by (8.2.8)

$$\begin{aligned} Y_2(x) &= [-3xI + o(x^{1-\theta})]P, \\ Y_2'(x) &= [-3I + o(x^{-\theta})]P. \end{aligned} \qquad (x \to \infty). \tag{8.2.11}$$

3. We now define

$$Y_1(x, \lambda_0) = Y_2(x, \lambda_0)\left[I - P + \int_0^x Y_2^*(t, \lambda_0)Y_2(t, \lambda_0)\,dt\right]^{-1} \tag{8.2.12}$$

and we consider the transformation

$$\begin{aligned} \Phi_1(x, \lambda) &= \Phi_2(x, \lambda) + \\ &+ Y_1(x, \lambda_0)W\{Y_2^*(x, \lambda_0), \ \Phi_2(x, \lambda)\}(\lambda^2 - \lambda_0^2)^{-1}. \end{aligned} \tag{8.2.13}$$

According to Theorem 6.2.1, it takes any solution $\Phi_2(x, \lambda)$ of (8.2.4) for $\lambda \neq \lambda_0$ into a solution $\Phi_1(x, \lambda)$ of

$$Y'' - V_1(x)Y + \lambda^2 Y = 0, \tag{8.2.14}$$

where

$$V_1(x) = V_2(x) - 2[Y_1(x, \lambda_0)Y_2^*(x, \lambda_0)]'. \tag{8.2.15}$$

From (8.2.6) and (8.2.12), it easily follows that $Y_2(x, \lambda_0)$, $Y_1(x, \lambda_0)$ and their derivatives decrease exponentially as $x \to \infty$. Thus (see (8.2.3)),

$$\int^{\infty} x^{1+\theta} |V_1(x)|\,dx < \infty \qquad (\theta < \varepsilon). \tag{8.2.16}$$

By making use of (8.2.7) and (8.2.12), we then find that for any $\theta < \varepsilon$

$$\left.\begin{aligned} Y_1(x, \lambda_0) &= [3\bar{\nu}^{-1}x^{-2}I + o(x^{-2+\theta})]P \\ Y_1'(x, \lambda_0) &= [-6\bar{\nu}^{-1}x^{-3}I + o(x^{-3+\theta})]P \end{aligned}\right\} \quad (x \to 0), \qquad (8.2.17)$$

$$2[Y_1(x, \lambda_0)Y_2^*(x, \lambda_0)]' = -6x^{-2}P + o(x^{-2+\theta}) \qquad (x \to 0)$$

and therefore (see (8.2.3) and (8.2.15)),

$$\int_0 x^{1-\theta} \,|\, V_1(x) - 6x^{-2}P \,|\, dx < \infty \qquad (\theta < \varepsilon). \qquad (8.2.18)$$

Let $U_2(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 0,\ 1, \ldots,\ p-1)$ be a complete set of normalized matrix eigenfunctions of the problem (8.2.4)—(0) for which a Parseval equality analogous to (7.4.18) holds and which have the following asymptotic behavior at infinity:

$$\begin{aligned} U_2(x, \lambda) &= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) \qquad (\lambda > 0), \\ U_2(x, \lambda_k) &= e^{-|\lambda_k|x}[M_k + o(1)] \qquad (k = 0,\ 1,\ \ldots, p-1). \end{aligned} \qquad (8.2.19)$$

Since $U_2(x, \lambda)$ and $Y_2(x, \lambda_0)$ are both $O(x)$ as $x \to 0$.

$$\lim_{x \to 0} W\{Y_2^*(x, \lambda_0),\ U_2(x, \lambda)\} = 0$$

and by Lemma 6.3.1, the transformation (8.2.13) applied to $U_2(x, \lambda)$ may be expressed as

$$U_1(x, \lambda) = U_2(x, \lambda) - Y_1(x, \lambda_0) \int_0^x Y_2^*(t, \lambda_0) U_2(t, \lambda)\, dt$$

(Cf. (7.4.21)).

By the use of this formula, the reasoning of § 4, Chapter VII can be repeated literally to show that $U_1(x, \lambda_0) = 0$ [3] and that the matrices $U_1(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 1,\ 2, \ldots,\ p-1)$ are solutions of (8.2.14) vanishing at $x = 0$ and having the same asymptotic form at infinity as the $U_2(x, \lambda)$; moreover, a Parseval equality such as (7.4.23) holds. Thus, the matrices $U_1(x, \lambda)$ comprise a complete set of normalized eigenfunctions of the problem (8.2.14)—(0).

4. We now let $Y_1(x)$ denote the solution of (8.2.14) for $\lambda = 0$ which is the image of $Y_2(x)$ under the transformation (8.2.13). From (8.2.7) and (8.2.10), it follows that

$$\lim_{x \to 0} W\{Y_2^*(x, \lambda_0),\ Y_2(x)\} = 0,$$

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[3] According to (8.2.6), $U_2(x, \lambda_0) = Y_2(x, \lambda_0)$.

so that by Lemma 6.3.1,

$$Y_1(x) = Y_2(x) - Y_1(x, \lambda_0) \int_0^x Y_2^*(t, \lambda_0) Y_2(t)\, dt.$$

By then applying (8.2.17), we find from this formula that

$$Y_1(x) = O(x)P, \quad Y_1'(x) = O(1)P \qquad (x \to 0). \tag{8.2.20}$$

This formula in conjunction with (8.2.11) also shows that $Y_1(x)$ has the same asymptotic behavior at infinity as $Y_2(x)$, i.e., for any $\theta < \varepsilon$

$$\left.\begin{aligned} Y_1(x) &= [-3xI + o(x^{1-\theta})]P, \\ Y_1'(x) &= [-3I + o(x^{-\theta})]P \end{aligned}\right\} \quad (x \to \infty). \tag{8.2.21}$$

But by (8.2.20),

$$\lim_{x \to 0} W\{Y_1^*(x), Y_1(x)\} = 0.$$

Therefore, if we define

$$Y(x) = Y_1(x) \left[I - (1+m^2)P - \int_0^x Y_1^*(t) Y_1(t)\, dt \right]^{-1}, \tag{8.2.22}$$

in which $mP = M_p$ (see condition (b) on the given data), then by Theorem 6.2.1, the transformation

$$\Phi(x, \lambda) = \Phi_1(x, \lambda) - \lambda^{-2} Y(x) W\{Y_1^*(x), \Phi_1(x, \lambda)\} \tag{8.2.23}$$

will take any solution $\Phi_1(x, \lambda)$ of equation (8.2.14) for $\lambda \neq 0$ into a solution $\Phi(x, \lambda)$ of

$$Y'' - \{V_1(x) + 2[Y(x) Y_1^*(x)]'\} Y + \lambda^2 Y = 0. \tag{8.2.24}$$

It is easy to show by the use of (8.2.20) and (8.2.21), that (8.2.22) leads to the relations

$$\begin{aligned} Y(x) &= O(x)P, \quad Y'(x) = O(1)P && (x \to 0), \\ Y(x) &= [x^{-2}I + o(x^{-2-\theta})]P, && \\ Y'(x) &= [-2x^{-3}I + o(x^{-3-\theta})]P && (x \to \infty) \end{aligned} \tag{8.2.25}$$

for any $\theta < \varepsilon$. As a consequence,

$$2[Y(x) Y_1^*(x)]' = O(x)P \qquad (x \to 0) \tag{8.2.26}$$

and

$$2[Y(x) Y_1^*(x)]' = 6x^{-2}P + o(x^{-2-\theta}) \qquad (x \to \infty). \tag{8.2.27}$$

If we write (8.2.24) in the form

$$Y'' - [V(x) + 6x^{-2}P]Y + \lambda^2 Y = 0, \tag{7.1.0}$$

where

$$V(x) = V_1(x) + 2[Y(x)Y_1^*(x)]' - 6x^{-2}P, \tag{8.2.28}$$

then by (8.2.16), (8.2.18), (8.2.26), and (8.2.27), we find that the Hermitian matrix $V(x)$ satisfies the condition (7.1.1).

We next observe that

$$\lim_{x\to 0} W\{Y_1^*(x),\ U_1(x, \lambda)\} = 0 \quad (\lambda > 0;\ \lambda = \lambda_k,\quad k = 1, 2, \ldots, p-1),$$

and therefore according to Lemma 6.3.1 the transformation (8.2.23) applied to $U_1(x, \lambda)$ may be written in the form

$$U(x, \lambda) = U_1(x, \lambda) + Y(x)\int_0^x Y_1^*(t)U_1(t, \lambda)\,dt. \tag{8.2.29}$$

An immediate consequence of this is that $U(0, \lambda) = 0$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 1, 2, \ldots, p-1$). Also, just as in the case of Lemma 7.3.4, it is possible to show that the transformation (8.2.23) does not effect the asymptotic form of solutions ($\lambda \neq 0$) at infinity, and therefore (Cf. (8.2.19))

$$\left.\begin{aligned} U(x, \lambda) &= e^{i\lambda x}I - e^{-i\lambda x}S(-\lambda) + o(1) && (\lambda > 0),\\ U(x, \lambda_k) &= e^{-|\lambda_k|x}[M_k + o(1)] && (k = 1, 2, \ldots, p-1). \end{aligned}\right\}\ (x \to \infty).$$

Finally, applying Theorem 6.3.1 in conjunction with the formulas (8.2.29), (8.2.22), and the Parseval equality for the matrices $U_1(x, \lambda)$, we obtain the Parseval equality

$$\int_0^\infty \Phi(x)\Phi^*(x)\,dx = \sum_{k=1}^{p} U(\Phi;\ \lambda_k)U^*(\Phi;\ \lambda_k) + $$

$$+\frac{1}{2\pi}\int_0^\infty U(\Phi;\ \lambda)U^*(\Phi;\ \lambda)\,d\lambda,$$

$$U(\Phi:\ \lambda) = \text{l.i.m.}\int_0^\infty \Phi(x)U(x, \lambda)\,dx$$

(see the derivation of (7.4.24)), where $\lambda_p = 0$ and $U(x, \lambda_p) = mY(x)$. By (8.2.25), $U(0, \lambda_p) = 0$ and as $x \to \infty$

$$U(x, \lambda_p) = x^{-2}[mP + o(1)] = x^{-2}[M_p + o(1)].$$

These results clearly show that the given quantities $S(\lambda)$, λ_k^2, and M_k ($k = 1, 2, \dots, p$) satisfying the conditions 1°, 2°, and (b) comprise the scattering data of the problem (7.1.0)—(0) with a potential $V(x)$ determined by (8.2.28).

§ 3. Case (c)

1. Consider any given unitary matrix $S(\lambda)$, negative numbers λ_k^2 and non-negative Hermitian matrices M_k ($k = 1, 2, \dots, p$) with the properties 1°, 2°, and (c).

Let

$$F_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [I - S(\lambda)] e^{i\lambda t} \, d\lambda,$$

$$F(t) \doteq \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k| t} + F_s(t).$$

According to (c), $n_1 = 1$, so that the equation

$$x(t) + \int_0^{\infty} x(\xi) F(t+\xi) \, d\xi = 0, \qquad 0 \leq t < \infty \tag{8.3.1}$$

has one independent solution with a Fourier transform $\tilde{x}(\lambda)$ satisfying the conditions

$$\lim_{x \to \infty} \lambda^2 \tilde{x}(\lambda) (I - P) = 0 \tag{8.3.2}$$

and

$$\tilde{x}(0) (I - P) \neq 0, \quad \text{if} \quad S(0) = I. \tag{8.3.3}$$

From (8.3.2) and the uniqueness of the solution of (8.3.1), it follows that $\tilde{x}(\lambda) P \not\equiv 0$ (see footnote 2 of this chapter), and therefore for some pure imaginary value of λ_0 ($\operatorname{Im} \lambda_0 < 0$) we shall have

$$\tilde{x}(\lambda_0) P \neq 0. \tag{8.3.4}$$

We may also assume that $\lambda_0^2 \neq \lambda_k^2$ ($k = 1, 2, \dots, p$).

We now take some positive quantity α such that $\alpha < |\lambda_k|$ ($k = 0, 1, \dots, p$), and we form the matrix

$$T(\lambda) = I - 2\alpha(\alpha + i\lambda)^{-1} Q, \tag{8.3.5}$$

where Q is tentatively any Hermitian projection matrix of rank one.

It can be verified directly that

$$T(\lambda)T(-\lambda) = T(-\lambda)T(\lambda) = I \qquad (\lambda \neq \pm i\alpha),$$
$$T(-\lambda) = T^*(\lambda) \qquad (\operatorname{Im} \lambda = 0),$$

and therefore for real values of λ, $T(\lambda)$ is a unitary matrix. The matrix

$$\hat{S}(\lambda) = T(\lambda)S(\lambda)T(\lambda) \qquad (-\infty < \lambda < \infty) \tag{8.3.6}$$

is also clearly unitary.

We now set

$$M_0 = P$$

and introduce the non-negative Hermitian matrices

$$\hat{M}_k = T(-\lambda_k)M_k U_k \qquad (k = 0, 1, \ldots, p) \tag{8.3.7}$$

determined by the polar decomposition[4]

$$T(-\lambda_k)M_k = \hat{M}_k U_k^* \quad (U_k U_k^* = I; \; k = 0, 1, \ldots, p).$$

$\hat{M}_k$ clearly has the same rank as M_k for all values of k.

Lemma 8.3.1: *The projection Q of rank one appearing in* (8.3.5) *may be so chosen that the aggregate of quantities $\hat{S}(\lambda)$, λ_k^2, and $\hat{M}_k$ ($k = 0, 1, \ldots, p$) possess the properties of the scattering data of the non-singular boundary-value problem. In this connection, the matrix*

$$\hat{F}_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [I - \hat{S}(\lambda)] e^{i\lambda t} \, d\lambda$$

will satisfy the inequality

$$\int_0^\infty t^{1+\theta} \, | \hat{F}_s'(t) | \, dt < \infty \qquad (-\varepsilon < \theta < \varepsilon). \tag{8.3.8}$$

Proof: Applying (8.3.5), (8.3.6), and the convolution theorem, we can express $\hat{F}_s(t)$ in terms of $F_s(t)$ by means of formulas corresponding to (7.6.3) and (7.6.4). A straightforward computation then shows that for any choice of Q, the matrix $\hat{S}(\lambda)$ has the properties I_s and II_s whenever $S(\lambda)$ does. Moreover, both $\hat{F}_s'(t)$ and $F_s'(t)$ satisfy (8.3.8). The other properties in which we are interested are related to the integral equations

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[4] See footnote 5 of Chapter VII on p. 206.

$$u(t)+\int_0^\infty u(\xi)\hat{F}(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty, \tag{8.3.9}$$

$$-v(t)+\int_{-\infty}^0 v(\xi)\hat{F}_s(t+\xi)\,d\xi = 0, \qquad -\infty < t \le 0, \tag{8.3.10}$$

$$w(t)+\int_0^\infty w(\xi)\hat{F}_s(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty, \tag{8.3.11}$$

in which

$$\hat{F}(t) = \sum_{k=0}^{p} \hat{M}_k^2 e^{-|\lambda_k|t} + \hat{F}_s(t).$$

It is necessary to choose Q in such a way that the first two equations have no non-trivial solutions and such that the number of linearly independent solutions of the third is equal to the sum of the ranks of the matrices $\hat{M}_0, \hat{M}_1, \ldots, \hat{M}_p$.

In the case (c), the equation

$$-y(t)+\int_{-\infty}^0 y(\xi)F_s(t+\xi)\,d\xi = 0, \qquad -\infty < t \le 0 \tag{8.3.12}$$

either has no non-trivial solution ($n_2 = 0$) or has one such solution ($n_2 = 1$).

These two cases must be treated separately.

(1) $n_2 = 0$. In this case, the number of linearly independent solutions of

$$z(t)+\int_0^\infty z(\xi)F_s(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty \tag{8.3.13}$$

is equal, by assumption, to the sum of the ranks of the matrices M_1 $M_2, \ldots, M_p$ ($n_3 = r$). It is also known (by the corollary to Lemma 3.4.1) that the non-trivial solution $x(t)$ of (8.3.1) is simultaneously a solution of (8.3.13), and its Fourier transform $\tilde{x}(\lambda)$ is such that $\tilde{x}(\lambda_k)M_k = 0$ for $k = 1, 2, \ldots, p$. Therefore, if $z^{(1)}(t)$, $z^{(2)}(t), \ldots, z^{(r)}(t)$ are linearly independent solutions of (8.3.13) and $\tilde{z}^{(1)}(\lambda)$, $\tilde{z}^{(2)}(\lambda), \ldots,$

$\tilde{z}^{(r)}(\lambda)$ their Fourier transforms, then

$$\tilde{x}(\lambda) = \sum_{j=1}^{r} \gamma_j \tilde{z}^{(j)}(\lambda)$$

and

$$\sum_{j=1}^{r} \gamma_j \tilde{z}^{(j)}(\lambda_k) M_k = 0 \qquad (k = 1,\ 2, \ldots, p).$$

where γ_j are certain numbers which are not all zero.

This implies that the homogeneous system

$$\sum_{j=1}^{r} c_j \tilde{z}^{(j)}(\lambda_k) M_k = 0 \qquad (k = 1,\ 2, \ldots, p) \qquad (8.3.14)$$

of r linear equations (since the sum of the ranks of $M_1, \ldots, M_p$ is r) in the r unknowns $c_1, c_2, \ldots, c_r$, has a non-trivial solution $\gamma_1, \gamma_2, \ldots, \gamma_r$. From the uniqueness of the solution of (8.3.1) easily follows the uniqueness of the solution of the system (8.3.14). This means that the rank of the system is $r-1$, i.e., one of the equations of the system is linearly dependent on the others. Without any loss of generality, we may assume that this equation is the one associated with M_p and that M_p is of rank one. This is obvious if all of the matrices M_k are of rank one; however, the stated assumption may always be satisfied. In fact, if any of the matrices M_k are non-singular, then it is merely necessary to represent it in the form $M_k = \alpha_1 Q_k^{(1)} + \alpha_2 Q_k^{(2)}$, where α_1 and α_2 are the eigenvalues of M_k and $Q_k^{(1)}$ and $Q_k^{(2)}$ are mutually orthogonal projecting matrices of rank one, and to count the corresponding eigenvalues λ_k^2 twice: once with the normalization matrix $\alpha_1 Q_k^{(1)}$ and once with the normalization matrix $\alpha_2 Q_k^{(2)}$.

Since by (8.3.4)

$$\tilde{x}(\lambda_0)P = \sum_{j=1}^{r} \gamma_j \tilde{z}^{(j)}(\lambda_0)P \neq 0$$

and since the equation of the system (8.3.14) associated with the matrix M_p is linearly dependent on the others, the system

$$\left\{ \begin{array}{ll} \displaystyle\sum_{j=1}^{r} c_j \tilde{z}^{(j)}(\lambda_k) M_k = 0 & \qquad (k=1,\ 2, \ldots, p-1), \\ \displaystyle\sum_{j=1}^{r} c_j \tilde{z}^{(j)}(\lambda_0) P = 0 & \end{array} \right.$$

has only the trivial solution.

But then it is easy to show that the aggregate of quantities $S(\lambda)$, λ_k^2 and M_k ($k = 0, 1, \ldots, p-1$) is the scattering data of some non-singular boundary-value problem. Therefore, $S(\lambda)$ can be factored into the form

$$S(\lambda) = E^{-1}(-\lambda)E(\lambda) = E^*(-\lambda)\,[E^*(\lambda)]^{-1}, \qquad (8.3.15)$$

where $E(\lambda) = E(0, \lambda)$ and $E(x, \lambda)$ is the special solution of the matrix differential equation associated with this boundary-value problem. Also, $E^{-1}(\lambda)$ is regular everywhere in the lower half-plane except at the points $\lambda_0, \lambda_1, \ldots, \lambda_{p-1}$ where it has simple poles with the residues $N_0, \ldots, N_{p-1}$ (see § 2, Chapter II); moreover, the sum of the ranks of $N_0, N_1, \ldots, N_{p-1}$ is equal to r. Finally, the matrix $E(\lambda_p)M_p$ is not zero, and $E(-i\alpha)$ is non-singular.[5] Therefore, $E^{-1}(-i\alpha)E(\lambda_p)M_p$ is also a non-vanishing matrix, and a vector $q = \{q_1, q_2\}$ exists such that $qE^{-1}(-i\alpha)E(\lambda_p)M_p \neq 0$.

Let the matrix Q be a projection onto the direction of q.

Then it is obvious that the relation

$$xQE^{-1}(-i\alpha)\,E\,(\lambda_p)M_p = 0$$

may be satisfied if and only if x is annihilated by Q, i.e., $xQ = 0$.

The matrix Q will be used to construct the matrix $T(\lambda)$ defined by (8.3.5).

From (8.3.6) and (8.3.15), it follows that

$$\hat{S}(\lambda) = T(\lambda)E^{-1}(-\lambda)E(\lambda)T(\lambda) = T(\lambda)E^*(-\lambda)\,[E^*(\lambda)]^{-1}T(\lambda).$$

That is

$$\hat{S}(\lambda) = \hat{E}^{-1}(-\lambda)\hat{E}(\lambda) = \hat{E}^*(-\lambda)\,[\hat{E}^*(\lambda)]^{-1},$$

where $\hat{E}(\lambda) = E(\lambda)T(\lambda)$ is regular in the lower half-plane, and $\hat{E}^{-1}(\lambda)$ is regular in the lower half-plane except for the points $\lambda_0, \lambda_1, \ldots, \lambda_{p-1}$, and $-i\alpha$ where it has simple poles. In the neighborhood of the real axis, $\hat{E}(\lambda)$ and $\hat{E}^{-1}(\lambda)$ behave like $E(\lambda)$ and $E^{-1}(\lambda)$, respectively.

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[5] If $E(\lambda_p)M_p = 0$, then the non-singular boundary-value problem in question would have λ_p^2 as an eigenvalue with M_p the corresponding normalization matrix, whereas in reality it is not an eigenvalue (even if λ_p^2 were an eigenvalue, the corresponding normalization matrix would be orthogonal to M_p). The det $E(-i\alpha)$ then does not vanish since $-i\alpha \neq \lambda_k$ ($k = 0, 1, \ldots, p-1$).

Hence, the remark 2 of Theorem 3.5.1 implies that the equation (8.3.10) has only the trivial solution and that the number of linearly independent solutions of (8.3.11) is equal to the sum of the ranks of the residues of the matrix $\hat{E}^{-1}(\lambda)$. But the residues $\hat{N}_0$, $\hat{N}_1, \ldots$, $\hat{N}_{p-1}$ of $\hat{E}^{-1}(\lambda)$ at the corresponding poles $\lambda_0, \lambda_1, \ldots, \lambda_{p-1}$ are related to the residues N_k of the matrix $E^{-1}(\lambda)$ by the obvious formula

$$\hat{N}_k = T(-\lambda_k)N_k, \qquad (k = 0, 1, \ldots, p-1).$$

Moreover, by (8.3.5), its residue at the pole $-i\alpha$ is given by $\hat{N} = -2i\alpha QE^{-1}(-i\alpha)$ and therefore $\hat{N}$ is of rank one.

Thus

$$\sum_{k=0}^{p-1} \operatorname{rank} \hat{N}_k + \operatorname{rank} \hat{N} = \sum_{k=0}^{p-1} \operatorname{rank} N_k + 1 = r + 1,$$

and since

$$\sum_{k=0}^{p} \operatorname{rank} \hat{M}_k = \sum_{k=1}^{p} \operatorname{rank} M_k + \operatorname{rank} \hat{M}_0 = r + 1,$$

the number of linearly independent solutions of (8.3.11) is equal to the sum of the ranks of $M_0, M_1, \ldots, M_p$.

We must still show that if Q is chosen as above, then the equation (8.3.9) has only the trivial solution.

According to the corollary to Lemma 3.4.1, any solution $u(t)$ of (8.3.9) belonging to $L^2_{(2)}(0, \infty)$ is simultaneously a solution of (8.3.11) and its Fourier transform $\tilde{u}(\lambda)$ satisfies the conditions

$$\tilde{u}(\lambda_k)\hat{M}_k = \tilde{u}(\lambda_k)T(-\lambda_k)M_k = 0 \qquad (k = 0, 1, \ldots, p). \qquad (8.3.16)$$

Inasmuch as $u(t)$ is a solution of (8.3.11), by the remark 2 of Theorem 3.5.1, we have

$$\tilde{u}(\lambda) = \sum_{k=1}^{p-1} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{u}(\lambda_k)\hat{N}_k \hat{E}(\lambda) - \frac{2i\alpha}{\lambda^2 + \alpha^2} \tilde{u}(-i\alpha)\hat{N}\hat{E}(\lambda) =$$

$$= \sum_{k=1}^{p-1} \frac{2\lambda_k}{\lambda^2 - \lambda_k^2} \tilde{u}(\lambda_k)T(-\lambda_k)N_k E(\lambda)T(\lambda) -$$

$$- \frac{4\alpha^2}{\lambda^2 + \alpha^2} \tilde{u}(-i\alpha)QE^{-1}(-i\alpha)E(\lambda)T(\lambda).$$

But (8.3.16) implies that

$$\tilde{u}(\lambda_k)T(-\lambda_k)N_k = 0, \qquad (k = 0, 1, \ldots, p-1)$$

(see footnote 20 of Chapter III). Therefore

$$\tilde{u}(\lambda) = -\frac{4\alpha^2}{\lambda^2+\alpha^2}\tilde{u}(-i\alpha)QE^{-1}(-i\alpha)E(\lambda)T(\lambda), \qquad (8.3.17)$$

and

$$\tilde{u}(\lambda_p)T(-\lambda_p)M_p = -\frac{4\alpha^2}{\lambda_p^2+\alpha^2}\tilde{u}(-i\alpha)QE^{-1}(-i\alpha)E(\lambda_p)M_p = 0,$$

or

$$\tilde{u}(-i\alpha)QE^{-1}(-i\alpha)E(\lambda_p)M_p = 0.$$

Because of our choice of Q, this last result shows that $\tilde{u}(-i\alpha)Q = 0$, and hence, from (8.3.17) we conclude that $\tilde{u}(\lambda) \equiv 0$. This means that the equation (8.3.9) has only the trivial solution.

Thus the lemma is proved for the case $n_2 = 0$.

(2) $n_2 = 1$. Let $\tilde{y}(\lambda)$ denote the Fourier transform of the nontrivial solution of equation (8.3.12), and let us show that $\tilde{y}(i\alpha) \neq 0$ for all positive values of α. Equation (8.3.12) is equivalent to

$$\tilde{y}(-\lambda)S(\lambda) = -\tilde{y}(\lambda) \qquad (-\infty < \lambda < \infty).$$

This equation is obviously satisfied by both $\tilde{y}(\lambda)$ and the vector function

$$\tilde{y}^{(1)}(\lambda) = (\lambda^2+\alpha^2)^{-1}\tilde{y}(\lambda).$$

Now if $\tilde{y}(i\alpha)$ were zero for some value of $\alpha > 0$, then both $\tilde{y}^{(1)}(\lambda)$ and $\tilde{y}(\lambda)$ would be regular in the lower half-plane. As a consequence, they would be the respective Fourier transforms of certain solutions of (8.3.12). Since $\tilde{y}(\lambda)$ and $\tilde{y}^{(1)}(\lambda)$ are clearly linearly independent, this would contradict the fact that this equation has a unique solution ($n_2 = 1$). Hence, $\tilde{y}(i\alpha) \neq 0$ and there exists a projection Q of rank one such that

$$\tilde{y}(i\alpha)Q \neq 0. \qquad (8.3.18)$$

We now take this matrix Q as the one defining $T(\lambda)$ in (8.3.5).

With this choice of Q, we can show that (8.3.10) has only the trivial solution. Suppose that $\tilde{v}(\lambda)$ is the Fourier transform of some solution of this equation. The vector function $\tilde{v}(\lambda)$ is regular in the upper half-plane and for real values of λ,

$$\tilde{v}(-\lambda)\hat{S}(\lambda) = -\tilde{v}(\lambda),$$

which by (8.3.6) can be expressed as

$$\tilde{v}(-\lambda)T(\lambda)S(\lambda) = -\tilde{v}(\lambda)T(-\lambda) \qquad (-\infty < \lambda < \infty).$$

Now according to (8.3.5), $\tilde{v}(\lambda)T(-\lambda)$ is also regular in the upper halfplane and from the last equation it then follows that $\tilde{v}(\lambda)T(-\lambda)$ is the Fourier transform of a solution of (8.3.12). By virtue of the uniqueness of this solution,

$$\tilde{v}(\lambda)T(-\lambda) = \gamma\tilde{y}(\lambda),$$

where γ is some constant, and as a result (see (8.3.5)),

$$\tilde{v}(\lambda) = \gamma\tilde{y}(\lambda)T(\lambda) = \gamma\tilde{y}(\lambda)\,[I-2\alpha(\alpha+i\lambda)^{-1}Q].$$

This and (8.3.18) show that if $\gamma \neq 0$, then $\tilde{v}(\lambda)$ will have a pole at the point $\lambda = i\alpha$, which is impossible. Thus $\gamma = 0$ and hence $\tilde{v}(\lambda) \equiv 0$. In other words, (8.3.10) has only the trivial solution.

But then in this event, $\hat{S}(\lambda)$ is the scattering matrix of some non-singular boundary-value problem (see Theorem 5.6.1) and can therefore be represented in the form

$$\hat{S}(\lambda) = \hat{E}^{-1}(-\lambda)\hat{E}(\lambda) = \hat{E}^{*}(-\lambda)\,[\hat{E}^{*}(\lambda)]^{-1}, \qquad (8.3.19)$$

where $\hat{E}(\lambda) = \hat{E}(0, \lambda)$ and $\hat{E}(x, \lambda)$ is the special solution of the matrix differential equation associated with this boundary-value problem. The matrix $\hat{E}^{-1}(\lambda)$ is regular everywhere in the lower halfplane except at a finite number of points $\nu_1, \nu_2, \ldots, \nu_l$ on the imaginary axis where it has simple poles and corresponding residues of $N_1, \ldots, N_l$.

From (8.3.6) and (8.3.19), we next have

$$S(\lambda) = T(-\lambda)\hat{E}^{*}(-\lambda)\,[\hat{E}^{*}(\lambda)]^{-1}T(-\lambda) =$$

$$= T(-\lambda)\hat{E}^{*}(-\lambda)\,[T(\lambda)\hat{E}^{*}(\lambda)]^{-1},$$

and from this and the existence of a non-trivial solution of (8.3.12) it can be shown that

$$\hat{N}_k\hat{E}(-i\alpha)Q = 0 \qquad (k = 1, 2, \ldots, l). \qquad (8.3.20)$$

The proof is exactly the same as that of the equation (7.6.13).

Let us now consider equation (8.3.11). The Fourier transform $\tilde{w}(\lambda)$ of any solution of it is regular in the lower halfplane and is such that $\tilde{w}(-\lambda)\hat{S}(\lambda) = \tilde{w}(\lambda)$ for real values of λ. By (8.3.19) and the remark of Theorem 3.5.1, we also have

$$\tilde{w}(\lambda) = \sum_{k=1}^{l} \frac{2\nu_k}{\lambda^2 - \nu_k^2}\,\tilde{w}(\nu_k)\hat{N}_k\hat{E}(\lambda). \qquad (8.3.21)$$

From this and (8.3.20) it then follows that the vector function $\tilde{w}(\lambda)T(-\lambda)$ is also regular in the lower halfplane, and moreover, satisfies the equation $\tilde{w}(-\lambda)T(\lambda)S(\lambda) = \tilde{w}(\lambda)T(-\lambda)$ $(-\infty < \lambda < \infty)$. In other words, it is the Fourier transform of some solution of (8.3.13). Conversely, if $\tilde{z}(\lambda)$ is the Fourier transform of some solution of (8.3.13), then it satisfies the relation $\tilde{z}(-\lambda)S(\lambda) = \tilde{z}(\lambda)$ $(-\infty < \lambda < \infty)$ or the relation $\tilde{z}(-\lambda)T(-\lambda)\hat{S}(\lambda) = \tilde{z}(\lambda)T(\lambda)$. But since $\tilde{z}(\lambda)\ T(\lambda)$ is regular together with $\tilde{z}(\lambda)$ in the lower halfplane (see (8.3.5)), it must be the Fourier transform of some solution of (8.3.11).

As a consequence, (8.3.11) and (8.3.13) have the same number of linearly independent solutions. Since the number of linearly independent solutions of (8.3.13) is, by assumption, one more than the sum of the ranks of $M_1, M_2, \ldots, M_p$, this shows that the number of linearly independent solutions of (8.3.11) is equal to

$$\sum_{k=1}^{p} \operatorname{rank} M_k + 1 = \sum_{k=1}^{p} \operatorname{rank} \hat{M}_k + 1 = \sum_{k=0}^{p} \operatorname{rank} \hat{M}_k$$

Hence, the quantities $\hat{S}(\lambda)$, λ_k^2, and $\hat{M}_k$ $(k = 0, 1, \ldots, p)$ have the property V.

Finally, consider the equation (8.3.9). Since any solution $u(t)$ of this equation is simultaneously a solution of (8.3.11), its Fourier transform $\tilde{u}(\lambda)$ satisfies the relation (8.3.21). Hence, according to (8.3.20), $\tilde{u}(\lambda)T(-\lambda)$ is regular in the lower halfplane. The corollary to Lemma 3.4.1 further yields

$$\tilde{u}(-\lambda)\hat{S}(\lambda) = \tilde{u}(\lambda) \quad (\operatorname{Im}\lambda = 0); \ \tilde{u}(\lambda_k)\hat{M}_k = 0 \quad (k = 0, 1, \ldots, p)$$

and hence (see (8.3.6) and (8.3.7))

$$\tilde{u}(-\lambda)T(\lambda)S(\lambda) = \tilde{u}(\lambda)T(\lambda) \qquad (\operatorname{Im}\lambda = 0);$$

$$\tilde{u}(\lambda_k)T(-\lambda_k)M_k = 0 \qquad (k = 0, 1, \ldots, p).$$

Taking into consideration the regularity of $\tilde{u}(\lambda)T(-\lambda)$ in the lower halfplane, we conclude from these relations that $\tilde{u}(\lambda)T(-\lambda)$ is the Fourier transform of a solution of (8.3.1) and moreover, such that $\tilde{u}(\lambda_0)T(-\lambda_0)M_0 = 0$, where $M_0 = P$. Hence, by virtue of the uniqueness of the solution of (8.3.1), we have $\tilde{u}(\lambda)T(-\lambda) = \beta\tilde{x}(\lambda)$ and $\beta\tilde{x}(\lambda_0)P = 0$. Since $\tilde{x}(\lambda_0)P \neq 0$ by (8.3.4), this implies that $\beta = 0$. This means that $\tilde{u}(\lambda) = \beta\tilde{x}(\lambda)T(\lambda) \equiv 0$, i.e., the equation (8.3.9) has only the trivial solution.

Hence, the lemma is proved for both $n_2 = 0$ and $n_2 = 1$.

2. Thus, $\hat{S}(\lambda)$, λ_k^2, and $\hat{M}_k$ ($k = 0, 1, \ldots, p$) comprise the scattering data for a non-singular boundary-value problem. This problem is described by some equation

$$Y'' - V_3(x)Y + \lambda^2 Y = 0, \qquad 0 < x < \infty \tag{8.3.22}$$

and the boundary condition $Y(0, \lambda) = 0$, where according to (8.3.8), the Hermitian matrix $V_3(x)$ satisfies the inequality

$$\int_0^\infty x^{1+\theta} \mid V_3(x) \mid dx < \infty \qquad (-\varepsilon < \theta < \varepsilon). \tag{8.3.23}$$

Let $E_3(x, \lambda)$ be the special solution of (8.3.22) and set $E_3(0, \lambda) = E_3(\lambda)$. Then in exactly the same way as in § 6 of Chapter VII (see formula (7.6.12)), we find that the following relation holds for the Fourier transform $\tilde{x}(\lambda)$ of a solution of (8.3.1):

$$\begin{aligned}\tilde{x}(\lambda) &= \frac{2\lambda_0}{\lambda^2 - \lambda_0^2} \tilde{x}(\lambda_0) T(\lambda_0) N_0 E_3(\lambda) T(-\lambda) = \\ &= (\lambda^2 - \lambda_0^2)^{-1} a N_0 E_3(\lambda) T(-\lambda).\end{aligned} \tag{8.3.24}$$

Here, N_0 is the residue of $E_3^{-1}(\lambda)$ at its pole λ_0, and a is any constant vector. From this last result and (8.3.2), we then find that

$$N_0(I - P) = 0. \tag{8.3.25}$$

Next, taking into account (8.3.5) and the regularity of $\tilde{x}(\lambda)$ in the lower halfplane, we conclude from (8.3.24) that $N_0 E_3(-i\alpha)Q = 0$, and since $N_0 P = N_0$,

$$PE_3(-i\alpha)Q = 0.$$

Therefore,

$$\begin{aligned}E_3(-i\alpha)QE_3^*(-i\alpha) &= (I - P)E_3(-i\alpha)QE_3^*(-i\alpha)(I - P) = \\ &= k^2(I - P)\end{aligned}$$

and hence,

$$Q = k^2 E_3^{-1}(-i\alpha)(I - P)[E_3^{-1}(-i\alpha)]^* = k^2 CC^*,$$

where

$$C = E_3^{-1}(-i\alpha)(I - P). \tag{8.3.26}$$

It is easy to show that $k^2 = c^{-1}$, where c is a positive constant defined by the equation

$$c(I - P) = C^* C, \tag{8.3.27}$$

and hence

$$Q = c^{-1}CC^*. \tag{8.3.28}$$

In fact, $Q = Q^2 = k^4CC^*CC^* = k^2cQ$, from which it follows that $k^2c = 1$ or $k^2 = c^{-1}$.

Consider the matrix

$$Y_3(x, -i\alpha) = E_3(x, -i\alpha)C, \tag{8.3.29}$$

which is a solution of (8.3.22) for $\lambda = -i\alpha$. From (8.3.26) and (8.3.29) we conclude on the basis of Theorems 1.2.1. and 1.4.1 that for any $\theta < \varepsilon$

$$\left.\begin{aligned} Y_3(x, -i\alpha) &= e^{-\alpha x}[C + o(x^{-\theta})], \\ Y_3'(x, -i\alpha) &= -\alpha e^{-\alpha x}[C + o(x^{-\theta})] \end{aligned}\right\} \quad (x \to \infty), \tag{8.3.30}$$

$$\left.\begin{aligned} Y_3(x, -i\alpha) &= I - P + o(x^{\theta}), \\ Y_3'(x, -i\alpha) &= o(x^{-1+\theta}) \end{aligned}\right\} \quad (x \to 0). \tag{8.3.31}$$

Now consider the matrix

$$Y_3(x, \lambda_0) = E_3(x, \lambda_0)N_0, \tag{8.3.32}$$

whose non-zero column is an eigenfunction of the problem (8.3.22)—(0) corresponding to the eigenvalue λ_0^2. According to (3.1.7), (3.1.8), and (8.3.25),

$$E_3'(0, \lambda_0)N_0 = 2\lambda_0 N_0^* A_0 N_0 = 2\lambda_0 P N_0^* A_0 N_0 P,$$

where

$$A_0 = \int_0^\infty E_3^*(x, \lambda_0)E_3(x, \lambda_0)\, dx,$$

and this shows that

$$Y_3'(0, \lambda_0) = E_3'(0, \lambda_0)N_0 = \nu P,$$

in which ν is a non-zero constant. This leads to the conclusion that

$$\left.\begin{aligned} Y_3(x, \lambda_0) &= [\nu x I + o(x^{1+\theta})]P, \\ Y_3'(x, \lambda_0) &= [\nu I + o(x^{\theta})]P \end{aligned}\right\} \quad (x \to 0) \tag{8.3.33}$$

(Cf. (8.2.7)).

Suppose that $G_3(x)$ is the solution of (8.3.22) for $\lambda = 0$ satisfying the conditions $G_3(0) = 0$ and $G_3'(0) = I$. Then according to Lemma 1.5.1,

$$\left.\begin{aligned} G_3(x) &= xE_3^*(0) + o(x^{1-\theta}), \\ G_3'(x) &= E_3^*(0) + o(x^{-\theta}) \end{aligned}\right\} \quad (x \to \infty). \tag{8.3.34}$$

for any $\theta < \varepsilon$.

LEMMA 8.3.2: *If* det $E_3(0)=0$, *then*

$$E_3(0)T(0)=E_3(0)T(0)P\neq 0.$$

This lemma is proved in exactly the same way as Lemma 8.2.1. In this connection, it is necessary to make use of the relation

$$E_3(\lambda) = E_3(-\lambda)\hat{S}(\lambda) \qquad (\lambda^2 > 0),$$

or (see (8.3.6))

$$E_3'(\lambda)T(-\lambda) = E_3(-\lambda)T(\lambda)S(\lambda) \qquad (\lambda^2 > 0),$$

and to take into consideration that $S(0)\neq I$ when det $E_3(0)=0$ (because in that case $\hat{S}(0)\neq I$).

LEMMA 8.3.3.: *There exists a matrix A such that*

$$E_3^*(0)A = -3T(0)P; \quad AP = A; \quad (I-P)A \neq 0.$$

If det $E_3(0)\neq 0$, *then A is uniquely determined by these conditions; on the other hand, if* det $E_3(0)=0$, *then there exists a one-parameter family of matrices A satisfying these conditions.*

Proof: Suppose that det $E_3(0)\neq 0$. Then the matrix

$$A = -3[E_3^*(0)]^{-1}T(0)P$$

clearly satisfies the first two conditions of the lemma. Furthermore, (8.3.3), (8.3.24), and (8.3.25) imply that

$$PE_3(0)T(0)(I-P) \neq 0,$$

and since $T(0)=T^*(0)=T^{-1}(0)$, it follows that

$$(I-P)A = -3(I-P)[E_3^*(0)]^{-1}T(0)P \neq 0.$$

That is, A also satisfies the third condition.

Suppose now that det $E_3(0)=0$. Then by Lemma 8.3.2,

$$E_3(0)T(0) = \begin{pmatrix} 0 & e_{12} \\ 0 & e_{22} \end{pmatrix}, \qquad |e_{12}|+|e_{22}|^2 \neq 0.$$

Thus, all the requirements are satisfied by the matrix

$$A = \begin{cases} \begin{pmatrix} 0 & -(3+\beta\bar{e}_{22})\bar{e}_{12}^{-1} \\ 0 & \beta \end{pmatrix}, & \text{if} \quad e_{12}\neq 0, \\ \begin{pmatrix} 0 & \beta \\ 0 & -3\bar{e}_{22}^{-1} \end{pmatrix}, & \text{if} \quad e_{12}=0, \end{cases}$$

where β is an arbitrary parameter.

Let A be a matrix for which the conditions of Lemma 8.3.3 hold and set

$$Y_3(x) = G_3(x)A. \tag{8.3.35}$$

Then according to (8.3.34),

$$\begin{aligned} Y_3(x) &= [-3xT(0) + o(x^{1-\theta})]P, \\ Y_3'(x) &= [-3T(0) + o(x^{-\theta})]P \end{aligned} \qquad (x \to \infty). \tag{8.3.36}$$

3. Now define

$$Y_2(x, -i\alpha) = \\ = Y_3(x, -i\alpha)\left[P + \int_x^\infty Y_3^*(t, -i\alpha)Y_3(t, -i\alpha)dt\right]^{-1}. \tag{8.3.37}$$

and consider the transformation

$$\begin{aligned} \Phi_2(x, \lambda) &= \Phi_3(x, \lambda) - \\ &- Y_2(x, -i\alpha)W\{Y_3^*(x, -i\alpha), \Phi_3(x, \lambda)\}(\lambda^2 + \alpha^2)^{-1}. \end{aligned} \tag{8.3.38}$$

By Theorem 6.2.1, it carries any solution $\Phi_3(x, \lambda)$ $(\lambda \neq -i\alpha)$ of equation (8.3.22) into a solution $\Phi_2(x, \lambda)$ of

$$Y'' - V_2(x)Y + \lambda^2 Y = 0, \tag{8.3.39}$$

where

$$V_2(x) = V_3(x) + 2[Y_2(x, -i\alpha)Y_3^*(x, -i\alpha)]'. \tag{8.3.40}$$

From (8.3.30), (8.3.31), and (8.3.37), we find that for $\theta < \varepsilon$

$$\begin{aligned} Y_2(x, -i\alpha) &= 2\alpha c^{-1}e^{\alpha x}[C + o(x^{-\theta})], \\ Y_2'(x, -i\alpha) &= 2\alpha^2 c^{-1}e^{\alpha x}[C + o(x^{-\theta})] \end{aligned} \qquad (x \to \infty) \tag{8.3.41}$$

and

$$\begin{aligned} Y_2(x, -i\alpha) &= Y_2(0, -i\alpha) + o(x^{\theta}), \\ Y_2'(x, -i\alpha) &= o(x^{-1+\theta}) \end{aligned} \qquad (x \to 0). \tag{8.3.42}$$

Moreover,

$$Y_2(0, -i\alpha) = (I - P)Y_2(0, -i\alpha) \neq 0.$$

Applying (8.3.31) and (8.3.42), we obtain

$$2[Y_2(x, -i\alpha)Y_3^*(x, -i\alpha)]' = o(x^{-1+\theta}) \qquad (x \to 0),$$

which, by (8.3.23) and (8.3.40), shows that

$$\int_0 x^{1-\theta} \,|\, V_2(x) \,|\, dx < \infty \qquad (\theta < \varepsilon).$$

In order to determine the behavior of $V_2(x)$ at infinity, it is necessary to sharpen (8.3.30). This is done by making use of Theorem 1.4.1 and by then carrying out computations similar to those used at the beginning of § 5 of Chapter VII, where the transformation inverse to (8.3.38) was investigated.

As a result, it is found that for any $\theta < \varepsilon$

$$\int^{\infty} x^{1+\theta} \,|\, V_2(x) \,|\, dx < \infty$$

and therefore $V_2(x)$ satisfies the inequality

$$\int_0^{\infty} x^{1+\theta} \,|\, V_2(x) \,|\, dx < \infty \qquad (-\varepsilon < \theta < \varepsilon). \tag{8.3.43}$$

Let $\hat{U}_3(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 0, 1, \ldots, p$) denote a complete set of normalized matrix eigenfunctions of the problem (8.3.22)—(0), so that Parseval's equality holds, and such that

$$\left.\begin{aligned} \hat{U}_3(x, \lambda) &= e^{i\lambda x} I - e^{-i\lambda x} \hat{S}(-\lambda) + o(1) \qquad (\lambda > 0), \\ \hat{U}_3(x, \lambda_k) &= e^{-|\lambda_k| x} [\hat{M}_k + o(1)] \qquad (k = 0, 1, \ldots, p). \end{aligned}\right\} (x \to \infty)$$

Since $Y_3(x, -i\alpha)$ decreases exponentially as $x \to \infty$, Lemma 6.3.1 can be used to express the transformation (8.3.38) as applied to $\hat{U}_3(x, \lambda)$ in the form

$$\hat{U}_2(x, \lambda) = \hat{U}_3(x, \lambda) - Y_2(x, -i\alpha) \int_x^{\infty} Y_3^*(t, -i\alpha) \hat{U}_3(t, \lambda)\, dt$$

(Cf. (7.5.24)). This result and Theorem 6.3.1 enable one to show, as in Chapter VII, that a Parseval equality also holds for the matrices $\hat{U}_2(x, \lambda)$ ($\lambda > 0$; $\lambda = \lambda_k$, $k = 0, 1, \ldots, p$). Furthermore, as in the proof of Lemma 7.5.4, we find that

$$\left.\begin{aligned} \hat{U}_2(x, \lambda) &= e^{i\lambda x} T(-\lambda) - e^{-i\lambda x} T(\lambda) \hat{S}(-\lambda) + o(1) \quad (\lambda > 0), \\ \hat{U}_2(x, \lambda_k) &= e^{-|\lambda_k| x} [T(\lambda_k) \hat{M}_k + o(1)] \qquad (k = 0, 1, \ldots, p). \end{aligned}\right\} (x \to \infty) \tag{8.3.44}$$

4. Now let $Y_2(x, \lambda_0)$ be the image of $Y_3(x, \lambda_0)$ under the transformation (8.3.38). Define

$$Y_1(x, \lambda_0) = Y_2(x, \lambda_0) \left[I - P + \int_0^x Y_2^*(t, \lambda_0) Y_2(t, \lambda_0)\, dt \right]^{-1} \tag{8.3.45}$$

and consider the transformation

$$\begin{aligned}\Phi_1(x, \lambda) &= \Phi_2(x, \lambda) + \\ &+ Y_1(x, \lambda_0) W\{Y_2^*(x, \lambda_0), \ \Phi_2(x, \lambda)\} (\lambda^2 - \lambda_0^2)^{-1}.\end{aligned} \tag{8.3.46}$$

According to Theorem 6.2.1, it takes any solution $\Phi_2(x, \lambda)$ $(\lambda \neq \lambda_0)$ of equation (8.3.39) into a solution $\Phi_1(x, \lambda)$ of

$$Y'' - V_1(x)Y + \lambda^2 Y = 0, \tag{8.3.47}$$

where

$$V_1(x) = V_2(x) - 2[Y_1(x, \lambda_0) Y_2^*(x, \lambda_0)]'. \tag{8.3.48}$$

It is easy to see that $Y_2(x, \lambda_0)$ and $Y_1(x, \lambda_0)$ and their derivatives decrease exponentially as $x \to \infty$ simultaneously with the matrix $Y_3(x, \lambda_0)$ of (8.3.32). As a consequence,

$$\int^{\infty} x^{1+\theta} \, | V_1(x) | \, dx < \infty \qquad (\theta < \varepsilon), \tag{8.4.49}$$

since the analogous inequality holds for $V_2(x)$.

Now by (8.3.31) and (8.3.33)

$$\lim_{x \to 0} W\{Y_3^*(x, \ -i\alpha), \ Y_3(x, \lambda_0)\} = 0$$

so that by Lemma 6.3.1, the transformation (8.3.38) applied to $Y_3(x, \lambda_0)$ can be expressed as

$$Y_2(x, \lambda_0) = Y_3(x, \lambda_0) + Y_2(x, \ -i\alpha) \int_0^x Y_3^*(t, \ -i\alpha) Y_3(t, \lambda_0) \, dt.$$

Upon also taking (8.3.42) into consideration, we find from this result that for $\theta < \varepsilon$

$$\left.\begin{aligned} Y_2(x, \lambda_0) &= [\nu x I + o(x^{1+\theta})]P, \\ Y_2'(x, \lambda_0) &= [\nu I + o(x^{\theta})]P \end{aligned}\right\} \quad (x \to 0). \tag{8.3.50}$$

This together with (8.3.45) shows that

$$Y_1(x, \lambda_0) = [3\bar{\nu}^{-1} x^{-2} I + o(x^{-2+\theta})]P \qquad (x \to 0). \tag{8.3.51}$$

Moreover, this relation may be differentiated.

Thus,

$$2[Y_1(x, \lambda_0) Y_2^*(x, \lambda_0)]' = -6x^{-2}P + o(x^{-2+\theta}) \qquad (x \to 0)$$

which implies that (see (8.3.43) and (8.3.48))

$$\int_0 x^{1-\theta} \, | V_1(x) - 6x^{-2}P | \, dx < \infty \qquad (\theta < \varepsilon). \tag{8.3.52}$$

The further investigation of the transformation (8.3.46) is a literal repetition of the portion of § 5 of Chapter VII beginning with the derivation of formula (7.5.30) and ending with that of formula (7.5.32). In this connection, account must be taken of the fact that (8.3.32) implies that $\hat{U}_3(x, \lambda_0)=Y_3(x, \lambda_0)\, B$ where B is some constant matrix; therefore both $\hat{Y}_2(x, \lambda_0)$ and $\hat{U}_2(x, \lambda_0)=Y_2(x, \lambda_0)B$ are annihilated by the transformation (8.3.46), i.e., $\hat{U}_1(x, \lambda_0)=0$.

In this way we can ascertain that the matrices $\hat{U}_1(x, \lambda)$ $(\lambda>0;$ $\lambda=\lambda_k$, $k=1, 2, \ldots, p)$ obtained from $\hat{U}_2(x, \lambda)$ by means of the transformation (8.3.46) have the same asymptotic form at infinity as the latter matrices. Moreover, they satisfy Parseval's equality and, as $x\to 0$, can be represented in the form

$$\hat{U}_1(x, \lambda) = (I-P)\hat{U}_3'(0, \lambda)x - Y_1(x, -i\alpha)\cdot A(\lambda) + o(x^{1+\theta}) \qquad (\theta<\varepsilon). \tag{8.3.53}$$

where

$$A(\lambda) = \int_0^\infty Y_3^*(t, -i\alpha)\hat{U}_3(t, \lambda)\, dt = (\lambda^2+\alpha^2)^{-1}Y_3^*(0, -i\alpha)\hat{U}_3'(0, \lambda)$$

and

$$Y_1(x, -i\alpha) = Y_2(x, -i\alpha) - Y_1(x, \lambda_0)\int_0^x Y_2^*(t, \lambda_0)Y_2(t, -i\alpha)\, dt \tag{8.3.54}$$

is a solution of equation (8.3.47) for $\lambda=-i\alpha$.

The proof of formula (8.3.53) is based on the following two properties of the solutions $\hat{U}_3(x, \lambda)$ of (8.3.22):

$$\hat{U}_3(0, \lambda)=0; \qquad \hat{U}_3(x, \lambda)=o(e^{\alpha x}) \qquad (x\to\infty).$$

But the solution $Y_3(x)$ of this equation for $\lambda=0$ also obviously has these two properties (see (8.3.35)). Therefore an expression analogous to (8.3.53) for $x\to 0$ holds for the solution $Y_1(x)$ of (8.3.47) obtained from $Y_3(x)$ by the successive transformations (8.3.38) and (8.3.46), i.e.,

$$\begin{gathered} Y_1(x) = (I-P)Y_3'(0)x - Y_1(x, -i\alpha)A(0) + o(x^{1+\theta}) \quad (\theta<\varepsilon), \\ A(0) = \alpha^{-2}Y_3^*(0, -i\alpha)Y_3'(0). \end{gathered} \tag{8.3.55}$$

From (8.3.42), (8.3.50), (8.3.51), and (8.3.54), it follows that

$$Y_1(x,\ -i\alpha) = Y_2(0,\ -i\alpha)+o(x^\theta) \qquad (\theta < \varepsilon) \qquad (8.3.56)$$

as $x \to 0$, and that this relation may be differentiated.

We next show that

$$W\{Y_1^*(x,\ -i\alpha),\ Y_1(x,\ -i\alpha)\} = 0. \qquad (8.3.57)$$

Since $Y_1(x,\ -i\alpha)$ satisfies equation (8.3.47) for $\lambda = -i\alpha$, the Wronskian of $Y_1^*(x,\ -i\alpha)$ and $Y_1(x,\ -i\alpha)$ is independent of x. But as $x \to \infty$, $Y_2(x,\ -i\alpha) = O(e^{\alpha x})$ by (8.3.41), whereas $Y_1(x,\ \lambda_0)$ and $Y_2(x,\ \lambda_0)$ are $O(e^{-|\lambda_0|x})$, with $\alpha < |\lambda_0|$.

Therefore, (8.3.54) yields $Y_1(x,\ -i\alpha) = Y_2(x,\ -i\alpha)+O(e^{-|\lambda_0|x})$ as $x \to \infty$, from which it follows that

$$W\{Y_1^*(x,\ -i\alpha),\ Y_1(x,\ -i\alpha)\} = \lim_{x\to\infty} W\{Y_1^*(x,\ -i\alpha),\ Y_1(x,\ -i\alpha)\} =$$
$$= \lim_{x\to\infty} W\{Y_2^*(x,\ -i\alpha),\ Y_2(x,\ -i\alpha)\}.$$

Now, by Lemma 6.2.2, the matrix $Y_2(x,\ -i\alpha)$ defined by (8.3.37) is such that

$$W\{Y_2^*(x,\ -i\alpha),\ Y_2(x,\ -i\lambda)\} \equiv 0,$$

and this leads to the required result (8.3.57).

5. On the basis of (8.3.55), (8.3.56), and (8.3.57),

$$\lim_{x\to 0} W\{Y_1^*(x),\ Y_1(x)\} = \alpha^{-2}Y_3^{*\prime}(0)\left[-Y_3(0,\ -i\alpha)Y_2^*(0,\ -i\alpha)(I-P)+\right.$$
$$\left.+(I-P)Y_2(0,\ -i\alpha)Y_3^*(0,\ -i\alpha)\right]Y_3'(0) = 0,$$

and since the Wronskian of $Y_1^*(x)$ and $Y_1(x)$ is independent of x, it follows that

$$W\{Y_1^*(x),\ Y_1(x)\} \equiv 0.$$

Therefore, if

$$Y(x) = Y_1(x)\left[I-(1+m^2)P-\int_0^x Y_1^*(t)Y_1(t)\,dt\right]^{-1}, \qquad (8.3.58)$$

where m^2 is a positive quantity to be determined below, then by Theorem 6.2.1, the transformation

$$\Phi(x,\ \lambda) = \Phi_1(x,\ \lambda)-\lambda^{-2}Y(x)W\{Y_1^*(x),\ \Phi_1(x,\ \lambda)\} \qquad (8.3.59)$$

takes any solution $\Phi_1(x,\ \lambda)$ $(\lambda \neq 0)$ of equation (8.3.47) into some solution $\Phi(x,\ \lambda)$ of

$$Y''-\{V_1(x)+2[Y(x)Y_1^*(x)]'\}Y+\lambda^2 Y = 0. \qquad (8.3.60)$$

We then find from (8.3.55), (8.3.56), and (8.3.58) that as $x \to 0$

$$Y_1(x) = Y_1(0) + o(x^{\theta}), \quad Y(x) = Y(0) + o(x^{\theta}) \qquad (\theta < \varepsilon),$$

and these relations may be differentiated.

As a consequence,

$$2[Y(x)Y_1^*(x)]' = o(x^{-1+\theta}) \qquad (x \to 0)$$

for any $\theta < \varepsilon$, from which we conclude by means of (8.3.52) that the Hermitian matrix

$$V(x) = V_1(x) + 2[Y(x)Y_1^*(x)]' - 6x^{-2}P \tag{8.3.61}$$

satisfies the condition

$$\int_0 x^{1-\theta} \, | V(x) | \, dx < \infty \qquad (\theta < \varepsilon). \tag{8.3.62}$$

To determine the behavior of $V(x)$ at infinity, it is necessary to know the asymptotic form of $Y_1(x)$ there. From (8.3.30), (8.3.36), (8.3.38), and (8.3.41) it follows that

$$\begin{aligned} Y_2(x) &= Y_3(x) - Y_2(x, -i\alpha)W\{Y_2^*(x, -i\alpha), Y_3(x)\}\alpha^{-2} = \\ &= [I - 2c^{-1}CC^* + o(x^{-\theta})][-3xT(0) + o(x^{1-\theta})]P - \qquad (x \to \infty) \\ &\quad - 2\alpha^{-1}c^{-1}[CC^* + o(x^{-\theta})][-3T(0) + o(x^{-\theta})]P. \end{aligned}$$

But since $I - 2c^{-1}CC^* = T(0) = T^{-1}(0)$ by (8.3.5) and (8.3.28), we obtain

$$Y_2(x) = [-3xI + o(x^{1-\theta})]P \qquad (x \to \infty;\ \theta < \varepsilon).$$

Now $Y_1(x)$ is the image of $Y_2(x)$ under the transformation (8.3.46) which does not effect the asymptotic form of the solutions at infinity. Therefore for any $\theta < \varepsilon$

$$Y_1(x) = [-3xI + o(x^{1-\theta})]P \qquad (x \to \infty) \tag{8.3.63}$$

and by (8.3.58),

$$Y(x) = [x^{-2}I + o(x^{-2-\theta})]P \qquad (x \to \infty).$$

Moreover, both of these relations may be differentiated.

Hence,

$$2[Y(x)Y_1^*(x)]' = 6x^{-2}P + o(x^{-2-\theta}) \qquad (x \to \infty),$$

from which we find, by (8.3.49) and (8.3.61), that

$$\int^{\infty} x^{1+\theta} \, | V(x) | \, dx < \infty \qquad (\theta < \varepsilon).$$

This result together with (8.3.62) leads to the inequality

$$\int_0^\infty x^{1+\theta}\,|\,V(x)\,|\,dx < \infty \qquad (-\varepsilon < \theta < \varepsilon).$$

Thus, as a consequence of the transformation (8.3.59), we have obtained the equation (8.3.60) which can be expressed as

$$Y'' - [V(x) + 6x^{-2}P]Y + \lambda^2 Y = 0, \tag{7.1.0}$$

where $V(x)$ is given by (8.3.61) and satisfies the conditon (7.1.1).

In order to show that the transformation (8.3.59) applied to $\hat{U}_1(x, \lambda)$ can be written in integral form, it is necessary to compute

$$\lim_{x\to 0} W\{Y_1^*(x),\ U_1(x, \lambda)\}.$$

To do this, we make use of (8.3.53), (8.3.55), (8.3.56), and (8.3.57) and we find that

$$\begin{aligned}
&\lim_{x\to 0} W\{Y_1^*(x),\ \hat{U}_1(x, \lambda)\} = \\
&= -A^*(0)Y_1^*(0, -i\alpha)(I-P)\hat{U}_3'(0, \lambda) + \\
&\quad + Y_3^{*\prime}(0)(I-P)Y_1(0, -i\alpha)A(\lambda),
\end{aligned}$$

or after the insertion of the values of $A(0)$ and $A(\lambda)$,

$$\begin{aligned}
&\lim_{x\to 0} W\{Y_1^*(x),\ \hat{U}_1(x, \lambda)\} = \\
&= Y_3^{*\prime}(0)[-\alpha^{-2}Y_3(0, -i\alpha)Y_1^*(0, -i\alpha)(I-P) + \\
&\quad + (\lambda^2+\alpha^2)^{-1}(I-P)Y_1(0, -i\alpha)Y_3^*(0, -i\alpha)]\hat{U}_3'(0, \lambda).
\end{aligned}$$

The formulas (8.3.31), (8.3.37), and (8.3.54) further yield

$$Y_3(0, -i\alpha) = I-P, \quad Y_1(0, -i\alpha) = (I-P)B,$$

where $B = \|\, b_{jk}\,\|_1^2$ is a positive definite Hermitian matrix. Thus

$$\begin{aligned}
&\lim_{x\to 0} W\{Y_1^*(x),\ \hat{U}_1(x, \lambda)\} = \\
&= Y_3^{*\prime}(0)[-\alpha^{-2}(I-P)B(I-P) + \\
&\quad + (\lambda^2+\alpha^2)^{-1}(I-P)B(I-P)]\hat{U}_3'(0, \lambda) = \\
&= -\lambda^2\alpha^{-2}(\lambda^2+\alpha^2)^{-1}b_{11}Y_3^{*\prime}(0)(I-P)\hat{U}_3(0, \lambda).
\end{aligned}$$

From the properties of B it clearly follows that b_{11} is positive. Finally, according to (8.3.53) and (8.3.55),

$$\begin{aligned} \hat{U}_1(0, \lambda) &= -(\lambda^2+\alpha^2)^{-1}Y_1(0, -i\alpha)Y_3^*(0, -i\alpha)\hat{U}_3'(0, \lambda) = \\ &= -(\lambda^2+\alpha^2)^{-1}(I-P)B(I-P)\hat{U}_3'(0, \lambda) = \\ &= -b_{11}(\lambda^2+\alpha^2)^{-1}(I-P)\hat{U}_3'(0, \lambda), \end{aligned} \tag{8.3.64}$$

$$\begin{aligned} Y_1(0) &= -\alpha^{-2}Y_1(0, -i\alpha)Y_3^*(0, -i\alpha)Y_3'(0) = \\ &= -\alpha^{-2}(I-P)B(I-P)Y_3'(0) = \\ &= -b_{11}\alpha^{-2}(I-P)Y_3'(0), \end{aligned} \tag{8.3.65}$$

and we finally obtain

$$\lim_{x\to 0} W\{Y_1^*(x), \hat{U}_1(x, \lambda)\} = -\lambda^2 b_{11}^{-1}Y_1^*(0)\hat{U}_1(0, \lambda). \tag{8.3.66}$$

By means of these formulas and Lemma 6.3.1, the transformation (8.3.59) applied to $\hat{U}_1(x, \lambda)$ can be written in the form

$$\begin{aligned} \hat{U}(x, \lambda) = \hat{U}_1(x, \lambda)+Y(x)\int_0^x Y_1^*(t)\hat{U}_1(t, \lambda)\,dt+ \\ +b_{11}^{-1}Y(x)Y_1^*(0)\hat{U}_1(0, \lambda). \end{aligned} \tag{8.3.67}$$

Hence,

$$\begin{aligned} \hat{U}(0, \lambda) &= [I+b_{11}^{-1}Y(0)Y_1^*(0)]\hat{U}_1(0, \lambda) = \\ &= [I+b_{11}^{-1}Y_1(0)(I-P-m^{-2}P)Y_1^*(0)]\hat{U}_1(0, \lambda) \end{aligned}$$

(see (8.3.58)). But from (8.3.64), (8.3.65), and the properties of $Y_3(x)$, it follows that

$$(I-P)Y_1(0)P = Y_1(0) = \| y_{jk}(0) \|_1^2 \neq 0, \tag{8.3.68}$$

$$(I-P)\hat{U}_1(0, \lambda) = \hat{U}_1(0, \lambda),$$

and so

$$\hat{U}(0, \lambda) = (1-b_{11}^{-1}m^{-2}|y_{12}(0)|^2)\hat{U}_1(0, \lambda) \qquad (y_{12}(0) \neq 0).$$

Therefore if we choose

$$m^2 = b_{11}^{-1}\,|\,y_{12}(0)\,|^2 \tag{8.3.6}$$

in formula (8.3.58), we shall have $\hat{U}(0, \lambda)=0$. In other words, any matrix $\hat{U}(x, \lambda)$ obtained from $\hat{U}_1(x, \lambda)$ by the transformation (8.3.59) will satisfy not only the equation (8.3.60) but also the boundary condition $Y(0, \lambda)=0$.

By means of the representation (8.3.67) analogous to (7.5.35), the very same reasoning of sub-section 5, § 5 of Chapter VII can be used to show that Parseval's equality holds for the matrices $\hat{U}(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 1, 2, \ldots, p)$. Since $\hat{U}(x, \lambda)$ is the image of $\hat{U}_2(x, \lambda)$ under the succesive transformations (8.3.46) and (8.3.59) which do not effect asymptotic behavior at infinity, the matrices $\hat{U}(x, \lambda)$ are asymptotically equal to the matrices appearing in (8.3.44).

Now, in exactly the same way as in sub-section 6, § 5, of Chapter VII, we may go from $\hat{U}(x, \lambda)$ to new matrices $U(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 1, 2, \ldots, p)$ also satisfying equation (8.3.60), vanishing at $x = 0$, yielding a corresponding Parseval equality, and having the following asymptotic form at infinity:

$$U(x, \lambda) = e^{i\lambda x} I - e^{-i\lambda x} S(-\lambda) + o(1) \qquad (\lambda > 0),$$
$$U(x, \lambda_k) = e^{-|\lambda_k| x}[M_k + o(1)] \qquad (k = 1, 2, \ldots, p).$$

Thus, we have shown that the given quantities $S(\lambda)$, λ_k^2, and M_k $(k = 1, 2, \ldots, p)$ consititute the scattering data for the boundary-value problem (7.1.0)—(0) with a potential $V(x)$ determined by (8.3.61).

Remark. Similarly, from $\hat{U}_1(x, \lambda)$, we may obtain matrices $U_1(x, \lambda)$ $(\lambda > 0;\ \lambda = \lambda_k,\ k = 1, 2, \ldots, p)$ which are solutions of (8.3.47), for which Parseval's equality holds, and which have the following behavior at infinity:

$$U_1(x, \lambda) = e^{i\lambda x} I - e^{-i\lambda x} S(-\lambda) + o(1) \qquad (\lambda > 0),$$
$$U_1(x, \lambda_k) = e^{-|\lambda_k| x}[M_k + o(1)] \qquad (k = 1, 2, \ldots, p).$$

Here, $U_1(x, \lambda) = \hat{U}_1(x, \lambda)\ T(\lambda)$ $(\lambda > 0)$ so that (8.3.66) yields

$$\lim_{x \to 0} W\{Y_1^*(x),\ U_1(x, \lambda)\} = -\lambda^2 b_{11}^{-1} Y_1^*(0) U_1(0, \lambda). \qquad (8.3.70)$$

§ 4. Algorithm for Determining the Potential Matrix. Examples

1. In consequence of the foregoing analysis of the boundary-value problem (7.1.0)—(0), we have deduced the following fundamental result.

In order for a unitary matrix $S(\lambda)$, numbers $\lambda_1^2 < \lambda_2^2 < \ldots < \lambda_p^2 \leq 0$ and non-negative Hermitian matrices $M_1, M_2, \ldots, M_p$ to be the scattering data for some problem (7.1.0)—(0) with a Hermitian potential $V(x)$ satisfying the condition (7.1.1), it is necessary and sufficient that the aggregate

$$S(\lambda),\ \lambda_k^2,\ M_k \qquad (k = 1, 2, \ldots, p) \tag{8.4.1}$$

have the properties 1°, 2°, and one of the properties (a), (b), or (c).

An analysis of the results of §§ 1—3 of this chapter also reveals an algorithm for computing the potential $V(x)$ from the given scattering data. Namely, in each of the three possible cases, there exists the formula (see (8.1.14), (8.2.28), and (8.3.61))

$$V(x) = V_1(x) + 2[Y(x)Y_1^*(x)]' - 6x^{-2}P. \tag{8.4.2}$$

Hence, it is merely necessary to show how the matrices $V_1(x)$, $Y_1(x)$, and $Y(x)$ may be found from the quantities (8.4.1).

The matrix $V_1(x)$ always satisfies the condition

$$\int^{\infty} x^{1+\theta} \, | V_1(x) | \, dx < \infty \qquad (\theta < \varepsilon)$$

and therefore, by Theorem 1.3.1 and its remark, the equation

$$Y'' - V_1(x)Y + \lambda^2 Y = 0 \tag{8.4.3}$$

has the special solution

$$E_1(x, \lambda) = e^{-i\lambda x} I + \int_x^{\infty} K_1(x, t) e^{-i\lambda t} \, dt, \tag{8.4.4}$$

and

$$V_1(x) = -2 \frac{d}{dx} K_1(x, x). \tag{8.4.5}$$

Furthermore, Parseval's equality holds for the matrix functions

$$\begin{aligned} U_1(x, \lambda) &= E_1(x, -\lambda) - E_1(x, \lambda) S(-\lambda) && (\lambda > 0), \\ U_1(x, \lambda_k) &= E_1(x, \lambda_k) M_k && (k = 1, 2, \ldots, p;\ \lambda_k \neq 0) \end{aligned} \tag{8.4.6}$$

(see (8.1.16), sub-section 3, § 2, and the remark at the end of § 3 of Chapter VIII). Hence, according to § 3 of Chapter IV, it follows that $K_1(x, y)$ is a solution of

$$K_1(x, y) + F(x + y) + \int_x^{\infty} K_1(x, t) F(t + y) \, dt = 0, \tag{8.4.7}$$

in which

$$F(t) = \sum_k{}' M_k^2 e^{-|\lambda_k| t} + \frac{1}{2\pi} \int_{-\infty}^{\infty} [I - S(\lambda)] e^{i\lambda t} \, d\lambda, \tag{8.4.8}$$

the summation extending over all non-zero eigenvalues in (8.4.1). We recall that by Theorem 3.4.1 the equation (8.4.7) has a unique solution $K_1(x, y)$ for any positive value of x.

Thus, after determining the matrix $F(t)$ from the scattering data (8.4.1) by means of the formula (8.4.8), we find $V_1(x)$ by first solving (8.4.7) and by then making use of (8.4.5).

As for the matrix $Y_1(x) = \| y_{jk} \|_1^2$, it is a solution of (8.4.3) for $\lambda=0$ and satisfies the following conditions (see (8.1.11), (8.2.20), (8.2.21), (8.3.63), and (8.3.69)):

(1) $Y_1(x)P=Y_1(x)$,

(2) $Y_1(x) = -3xP+o(x^{1-\theta})$ $(x \to \infty)$,

(3) $Y_1(0) = 0$ and $(I-P)\,Y_1'(0) = 0$ in the case (a);
$Y_1(0) = 0$ in the case (b);
$PY_1(0) = 0$, $(I-P)\,Y_1(0) \neq 0$ in the case (c).

The results of §§ 1–3 indicate that $Y_1(x)$ is uniquely determined by these conditions if $S(0) = I$; on the other hand if $S(0) \neq I$, then $Y_1(x)$ is uniquely determined only in the case (a) and in the other two cases, the expression for $Y_1(x)$ will contain an arbitrary parameter. We further note that if

$$\int^{\infty} x^2\,|\,V_1(x)\,|\,dx < \infty.$$

then $Y_1(x)$ may be found from the formula

$$Y_1(x) = \begin{pmatrix} 0 & \alpha \\ 0 & -3x+\beta \end{pmatrix} + \int_x^{\infty} K_1(x, t) \begin{pmatrix} 0 & \alpha \\ 0 & -3t+\beta \end{pmatrix} dt, \tag{8.4.9}$$

where α and β are determined by the conditions at $x = 0$.

Knowing $Y_1(x)$, we can then compute

$$Y(x) = Y_1(x) \left[I-(1+m^2)P - \int_0^x Y_1^*(t) Y_1(t)\,dt \right]^{-1} \tag{8.4.10}$$

(see (8.1.8), (8.2.22), and (8.4.58)). In the case (a), m^2 has the value 0, in the case (b), we take its value from the relation $M_p = mP$, and in the case (c)

$$m^2 = b_{11}^{-1}\,|\,y_{12}(0)\,|^2 \tag{8.4.11}$$

(see (8.3.69)), where b_{11}^{-1} is defined by the formula

$$\lim_{x\to 0} W\{Y_1^*(x),\ U_1(x,\lambda)\} = -\lambda^2 b_{11}^{-1} Y_1^*(0) U_1(0,\lambda) \quad (8.4.12))$$

(see (8.3.70)).

By then substituting the matrices $V_1(x)$, $Y_1(x)$, and $Y(x)$ thus obtained in the right-hand side of (8.4.2), we arrive at our desired potential $V(x)$. In this connection, if $S(0) = I$, then both $V(x)$ and $Y_1(x)$ will be uniquely determined by the scattering data. However, when $S(0) \neq 0$, $V(x)$ is uniquely determined only in the case (a). In the other two cases there exists a one-parameter family of Hermitian matrices $V(x)$ satisfying the condition (7.1.1) and such that the boundary-value problems (7.1.0)—(0) corresponding to them have the same scattering data.

2. In conclusion, we give several examples illustrative of all the possible cases.

Example 1. Suppose that $S(\lambda) = I$ and that there is no point spectrum.

Then

$$F_s(t) = F(t) \equiv 0$$

and conditions 1°, 2°, and (a) are clearly satisfied.

From (8.4.7), we find that $K_1(x, y) \equiv 0$ and therefore by (8.4.5)

$$V_1(x) \equiv 0.$$

Now applying (8.4.9), and (8.4.10) and taking into consideration the conditions on $Y_1(x)$ at $x = 0$, we obtain

$$Y_1(x) = -3xP,$$

$$Y(x) = -3xP[I-P-3x^3P]^{-1} = x^{-2}P$$

and thus by (8.4.2)

$$V(x) = V_1(x) + 2[Y(x)Y_1^*(x)]' - 6x^{-2}P \equiv 0.$$

Example 2. Let

$$S(\lambda) = \frac{\lambda+i}{\lambda-i}(I-P)+P$$

and suppose that there is no point spectrum (note that $S(0) \neq I$) Since $I-S(\lambda) = -2i(\lambda-i)^{-1}(I-P)$, it is easy to verify that

$$F_s(t) = \begin{cases} 2e^{-t}(I-P), & \text{if } t > 0 \\ 0\ , & \text{if } t < 0 \end{cases}$$

which, in turn, implies that

$$F(t) = 2e^{-t}(I-P) \qquad (t > 0).$$

A direct verification shows that the properties 1°, 2°, and (a) hold here just as in the preceding example.

Solving the equation (8.4.7), we obtain

$$K_1(x, y) = -e^{-y}(\cosh x)^{-1}(I-P),$$

so that

$$V_1(x) = -2\frac{d}{dx}K_1(x, x) = -2(\cosh x)^{-2}(I-P).$$

The formula (8.4.9) then yields

$$Y_1(x) = \begin{pmatrix} 0 & \alpha(1-e^{-2x})(1+e^{-2x})^{-1} \\ 0 & -3x+\beta \end{pmatrix},$$

and since, in the present case, $Y_1(0) = (I-P)Y_1'(0) = 0$, it follows that $\alpha = \beta = 0$. As a result,

$$Y_1(x) = -3xP.$$

Finally, by (8.4.10), we have

$$Y(x) = -3xP[I-P-3x^3P]^{-1} = x^{-2}P.$$

Substituting the expressions found above in (8.4.2), we arrive at

$$V(x) \equiv V_1(x) = -2(\cosh x)^{-2}(I-P).$$

Example 3. Let

$$S(\lambda) = (I-P)+\frac{(\lambda+i)(\lambda+2i)}{(\lambda-i)(\lambda-2i)}P; \; \lambda_1^2 = 0, \; M_1 = mP \qquad (m > 0).$$

Then

$$I-S(\lambda) = -6i\lambda(\lambda-i)^{-1}(\lambda-2i)^{-1}P,$$

$$F_s(t) = \begin{cases} (12e^{-2t}-6e^{-t})P, & \text{if} \quad t > 0 \\ \qquad 0 \qquad , & \text{if} \quad t < 0, \end{cases}$$

$$F(t) = (12e^{-2t}-6e^{-t})P \qquad (t > 0),$$

and it is not difficult to show that the conditions 1°, 2°, and (b) hold: the equation

$$x(t)+\int_0^\infty x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty$$

has the vector solution

$$x(t) = \{0,\ e^{-t} - e^{-2t}\}$$

which is unique to within a scalar factor.

Solving (8.4.7) and then making use of (8.4.5), (8.4.9), (8.4.10) and the condition $Y_1(0) = 0$, we find that

$$K_1(x,\ y) = 3e^{-y}(\cosh x - e^{-y})(\sinh x)^{-2}P;$$
$$V_1(x) = 6(\sinh x)^{-2}P;$$
$$Y_1(x) = \varphi(x)P,$$

where

$$\varphi(x) = -3x + \frac{9}{2} - \frac{9}{4}(e^{-2x} - 1 + 2x)(\sinh x)^{-2}; \qquad (8.4.13)$$

$$Y(x) = \varphi(x)P\left[I - (1+m^2)P - \int_0^x \varphi^2(t)\,dt\,P\right]^{-1} =$$
$$= -\varphi(x)\left[m^2 + \int_0^x \varphi^2(t)\,dt\right]^{-1} P.$$

One can easily show that the Hermitian matrix $V(x)$ obtained from the above quantities by means of (8.4.2) satisfies the condition (7.1.1).

Example 4. Suppose that

$$S(\lambda) = \frac{\lambda+i}{\lambda-i}(I-P) + \frac{(\lambda+i)(\lambda+2i)}{(\lambda-i)(\lambda-2i)}P;\ \lambda_1^2 = 0,\ M_1 = mP$$
$$(m > 0).$$

In this case

$$F_s(t) = \begin{cases} 2e^{-t}(I-P) + (12e^{-2t} - 6e^{-t})P, & \text{if} \quad t > 0 \\ 0\qquad\qquad\qquad\quad, & \text{if} \quad t < 0, \end{cases}$$
$$F(t) = 2e^{-t}(I-P) + (12e^{-2t} - 6e^{-t})P \qquad (t > 0)$$

and as in the preceding example, it is possible to show that the given quantities have the properties 1°, 2°, and (b). We note, however, that the distinction here is that $S(0) \neq I$.

The equation (8.4.7) and the formulas (8.4.5), (8.4.9), and (8.4.10) yield (Cf. Examples 2 and 3)

$$K_1(x, y) = -e^{-y}(\cosh x)^{-1}(I-P)+3e^{-y}(\cosh x - e^{-y})(\sinh x)^{-2}P;$$

$$V_1(x) = -2(\cosh x)^{-2}(I-P)+6(\sinh x)^{-2}P;$$

$$Y_1(x) = \begin{pmatrix} 0 & \alpha \tanh x \\ 0 & \varphi(x) \end{pmatrix},$$

where α is an arbitrary constant, and the function φ is the same as the one in Example 3 and given by (8.4.13);

$$Y(x) = \begin{pmatrix} 0 & -\alpha \tanh x \left[m^2 + \int\limits_0^x \{|\alpha|^2 \tanh^2 t + \varphi^2(t)\}\, dt \right]^{-1} \\ 0 & -\varphi(x) \left[m^2 + \int\limits_0^x \{|\alpha|^2 \tanh^2 t + \varphi^2(t)\}\, dt \right]^{-1} \end{pmatrix}.$$

It is clear that the arbitrary parameter α, on which both $Y_1(x)$ and $Y(x)$ depend, will enter into the expression for $V(x)$.

Example 5. Let $S(\lambda)$ be the same as in Example 4, and let $\lambda_1^2 = -1$ and $M_2 = I-P$.

These quantities have the properties 1°, 2°, and (c), and moreover, $n_2 = 0$. To satisfy ourselves that this is so, it suffices to observe that here $F_s(t)$ is the same as in Example 4, that

$$F(t) = 3e^{-t}(I-P)+(12e^{-2t}-6e^{-t})P \qquad (t > 0)$$

and that the equation

$$x(t) + \int\limits_0^\infty x(\xi)F(t+\xi)\, d\xi = 0, \qquad 0 \leq t < \infty$$

has the vector solution

$$x(t) = \{0,\ e^{-t}-e^{-2t}\}.$$

which is unique to within a scalar factor.

Now solving (8.4.7), we obtain

$$K_1(x, y) = -3e^{-(x+y)}\left(1+\frac{3}{2}e^{-2x}\right)^{-1}(I-P)+$$
$$+3e^{-y}(\cosh x - e^{-y})(\sinh x)^{-2}P.$$

By (8.4.5), (8.4.9), and the conditions on $Y_1(x)$ at $x = 0$, we find from this that

$$V_1(x) = -12e^{-2x}\left(1+\frac{3}{2}e^{-2x}\right)^{-2}(I-P)+6(\sinh x)^{-2}P;$$

$$Y_1(x)=\begin{pmatrix}0 & \alpha\psi(x)\\ 0 & \varphi(x)\end{pmatrix},$$

where α is an arbitrary constant distinct from zero,

$$\psi(x) = \left(1-\frac{3}{2}e^{-2x}\right)\left(1+\frac{3}{2}e^{-2x}\right)^{-1},$$

and $\varphi(x)$ is given by (8.4.13).

From (8.4.4), (8.4.6), (8.4.11) and (8.4.12), it then follows that

$$m^2=\frac{1}{5}|\alpha|^2,$$

and therefore by (8.4.10)

$$Y(x) = \begin{pmatrix}0 & -\alpha\psi(x)\left[\frac{1}{5}|\alpha|^2+\int\limits_0^x\{|\alpha|^2\psi^2(t)+\varphi^2(t)\}\,dt\right]^{-1}\\ 0 & -\varphi(x)\left[\frac{1}{5}|\alpha|^2+\int\limits_0^x\{|\alpha|^2\psi^2(t)+\varphi^2(t)\}\,dt\right]^{-1}\end{pmatrix}.$$

Substituting the results obtained in (8.4.2), we arrive at a one-parameter family of potentials $V(x)$ depending on the parameter α.

Example 6. Let

$$S(\lambda) = \begin{pmatrix}\dfrac{(\lambda+0.5i)^2(\lambda^2-0.75)}{(\lambda-0.5i)^3(\lambda-1.5i)} & \dfrac{2i\lambda(\lambda+0.5i)}{(\lambda-0.5i)^2(\lambda-1.5i)}\\[2ex] \dfrac{2i\lambda(\lambda+0.5i)}{(\lambda-0.5i)^2(\lambda-1.5i)} & \dfrac{\lambda^2-0.75}{(\lambda-0.5i)(\lambda-1.5i)}\end{pmatrix};$$

$$\lambda_1^2 = -\frac{9}{4},\quad \lambda_2^2 = -\frac{1}{4};\quad M_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}4 & 2\\ 2 & 1\end{pmatrix},\quad M_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

In this case

$$F_s(t) = \begin{cases}\begin{pmatrix}12e^{-\frac{3}{2}t}-\frac{1}{2}(t-4)^2e^{-\frac{t}{2}} & 6e^{-\frac{3}{2}t}+(t-4)e^{-\frac{t}{2}}\\ 6e^{-\frac{3}{2}t}+(t-4)e^{-\frac{t}{2}} & 3e^{-\frac{3}{2}t}-e^{-\frac{t}{2}}\end{pmatrix}, & t > 0,\\ 0, & t < 0,\end{cases}$$

$$F(t) = F_s(t)+M_1^2e^{-\frac{3}{2}t}+M_2^2e^{-\frac{t}{2}}\qquad (t > 0).$$

Consequently, $S(\lambda)$ being a unitary matrix, as one can easily verify, satisfies the conditions 1° and 2°.

It is further possible to show that the equation

$$z(t)+\int_0^\infty z(\xi)F_s(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty$$

has two linearly independent vector solutions

$$z^{(1)}(t) = \left\{t^2e^{-\frac{t}{2}},\ \ -2te^{-\frac{t}{2}}\right\};$$

$$z^{(2)}(t) = \left\{(t-2)e^{-\frac{t}{2}}+2e^{-\frac{3}{2}t},\ \ e^{-\frac{3}{2}t}-e^{-\frac{t}{2}}\right\},$$

and that the equation

$$x(t)+\int_0^\infty x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \leq t < \infty$$

has the unique solution (to within a multiplicative factor)

$$x(t) = \left\{t^2e^{-\frac{t}{2}},\ \ -2te^{-\frac{t}{2}}\right\}.$$

Moreover, the Fourier transform of the latter solution is

$$\tilde{x}(\lambda) = \left\{2i\left(\lambda-\frac{1}{2}i\right)^{-3},\ 2\left(\lambda-\frac{1}{2}i\right)^{-2}\right\}.$$

Since the equation

$$-y(t)+\int_{-\infty}^0 y(\xi)F_s(t-\xi)\,d\xi = 0, \qquad -\infty < t \leq 0$$

has only the trivial solution, the above results show that the given quantities also have the property (c) (with $n_2 = 0$), and therefore constitute the scattering data for some boundary-value problem (7.1.0)—(0). The corresponding potential $V(x)$ may be found by means of the formulas of sub-section 1. However, we shall not carry out the computations in view of their complexity.

Example 7. Let

$$S(\lambda) = \frac{\lambda-i}{\lambda+i}(I-P)+\frac{(\lambda+i)\,(\lambda+2i)}{(\lambda-i)\,(\lambda-2i)}\,P$$

and suppose that there is no point spectrum. Then

$$F_s(t) = \begin{cases} (12e^{-2t}-6e^{-t})P, & \text{if} \quad t > 0 \\ 2e^{t}(I-P), & \text{if} \quad t < 0, \end{cases}$$

$$F(t) = (12e^{-2t}-6e^{-t})P \qquad (t > 0)$$

and it is not difficult to verify that the properties 1°, 2°, and (c), with $n_2 = 1$, are here fulfilled.

Making use of the appropriate formulas of sub-section 1, we obtain

$$K_1(x, y) = 3e^{-y}(\cosh x - e^{-y})(\sinh x)^{-2}P;$$

$$V_1(x) = 6(\sinh x)^{-2}P;$$

$$Y_1(x) = \begin{pmatrix} 0 & \alpha \\ 0 & \varphi(x) \end{pmatrix},$$

where α is any non-vanishing constant, and $\varphi(x)$ is given by (8.4.13);

$$m^2 = |\alpha|^2,$$

$$Y(x) = \begin{pmatrix} 0 & -\alpha\left[|\alpha|^2+\int\limits_0^x \{|\alpha|^2+\varphi^2(t)\}\,dt\right]^{-1} \\ 0 & -\varphi(x)\left[|\alpha|^2+\int\limits_0^x \{|\alpha|^2+\varphi^2(t)\}\,dt\right]^{-1} \end{pmatrix}.$$

For the present example, $S(0) \neq I$, and in accordance with this, we have a one-parameter family of potentials $V(x)$ given by the formula (8.4.2).

Example 8. Suppose that there is no point spectrum and that

$$S(\lambda) = T(-\lambda)E^{-1}(-\lambda)E(\lambda)T(\ \lambda),$$

where

$$E(\lambda) = \begin{pmatrix} 1+\dfrac{\delta}{(\lambda-i)^4} & \dfrac{\delta(\lambda+i)(\lambda+2i)}{(\lambda-i)^4} \\ \dfrac{\delta(\lambda^2+4)}{(\lambda-i)^4} & \dfrac{\lambda^2+4}{(\lambda-i)(\lambda-2i)}\left[1+\dfrac{\delta}{(\lambda-i)^4}\right] \end{pmatrix} \qquad (\delta < 0),$$

$$T(\lambda) = I+2i(\lambda-i)^{-1}Q,$$

$$Q = \frac{1}{\left(1+\dfrac{\delta}{2^4}\right)^2+\left(\dfrac{6\delta}{2^4}\right)^2}\begin{pmatrix} \left(1+\dfrac{\delta}{2^4}\right)^2 & 6\left(1+\dfrac{\delta}{2^4}\right)\dfrac{\delta}{2^4} \\ 6\left(1+\dfrac{\delta}{2^4}\right)\dfrac{\delta}{2^4} & \left(\dfrac{6\delta}{2^4}\right)^2 \end{pmatrix}.$$

It is easily seen that $S(\lambda)$ is unitary and has the properties 1° and 2°. Moreover, $S(0) = I$. It is possible to also show that if δ is a sufficiently small positive quantity, then the condition (c), with $n_2 = 1$, will hold.

APPENDIX I

On the Characteristic Properties of the Scattering Data of the Boundary-Value Problem without Singularities

The scattering data of the non-singular boundary-value problem is characterized by the five properties formulated at the beginning of Chapter V. The examples of § 7 of that Chapter show that these properties are independent. However, it is possible to replace them by another set of characteristic properties. This result will make up the content of Theorem 2 which concludes this appendix.

§ 1. Factorization of a Unitary Matrix

LEMMA 1. *If for* $-\infty < \lambda < \infty, S(\lambda)$ *is a unitary matrix with the property* I_s, *then for any* $a > 0$, *the* matrix

$$S_a(\lambda) = S(\lambda)\frac{\lambda+ia}{\lambda-ia}$$

also has this property.

If, in addition, $S(\lambda)$ *has the property* II_s *then so will* $S_a(\lambda)$.

Proof: If $-\infty < \lambda < \infty$, then $S_a(\lambda)$ obviously is a unitary matrix together with $S(\lambda)$. Now, since

$$I-S_a(\lambda) = I-S(\lambda)+\frac{2a}{i\lambda+a}I-\frac{2a}{i\lambda+a}[I-S(\lambda)]$$

and

$$\frac{1}{i\lambda+a} = \int_0^\infty e^{-at}e^{-i\lambda t}\,dt,$$

we find by means of theorems on the convolution that $I—S_a(\lambda)$ is the Fourier transform of

$$F_{s,a}(t) = \begin{cases} F_s(t) + 2ae^{-at}I - 2a\Phi(t), & \text{if} \quad t > 0 \\ F_s(t) - 2a\Phi(t), & \text{if} \quad t < 0, \end{cases}$$

where $F_s(t)$ is the matrix having $I—S(\lambda)$ as its Fourier transform and

$$\Phi(t) = \int_0^\infty F_s(t-\xi)e^{-a\xi}\,d\xi = e^{-at}\int_{-\infty}^t F_s(u)e^{au}\,du. \qquad (1)$$

It is apparent that $F_{s,a}(t)$ is Hermitian along with $F_s(t)$. To complete the proof of the first part of the lemma, we must still show that the rows of $\Phi(t)$ belong to both $\hat{L}_{(n)}(-\infty, \infty)$ and $L^1_{(n)}(0, \infty)$, provided that the rows of $F_s(t)$ also have this property. The fact that the rows of $\Phi(t)$ belong to $\hat{L}_{(n)}(-\infty, \infty)$ is an immediate consequence of (1) and theorems on the convolution. Further,

$$\int_0^\infty |\Phi(t)|\,dt \leq \int_0^\infty e^{-at}\,dt \int_{-\infty}^t |F_s(u)|\,e^{au}\,du =$$

$$= \int_{-\infty}^0 |F_s(u)|\,e^{au}\,du \int_0^\infty e^{-at}\,dt + \int_0^\infty |F_s(u)|\,e^{au}\,du \int_u^\infty e^{-at}\,dt =$$

$$= \frac{1}{a}\left\{ \int_{-\infty}^0 |F_s(u)|\,e^{au}\,du + \int_0^\infty |F_s(u)|\,du \right\}$$

and hence by the properties of $F_s(t)$,

$$\int_0^\infty |\Phi(t)|\,dt < \infty.$$

Suppose now that $S(\lambda)$ has both the properties I_s and II_s. Then $S_a(\lambda)$ also has the property II_s, i.e., for all positive values of t, $F'_{s,a}(t)$ exists and

$$\int_0^\infty t\,|F'_{s,a}(t)|\,dt < \infty.$$

It clearly suffices to prove the statment for $\Phi(t)$.

From (1), we have for $t > 0$

$$\Phi'(t) = -ae^{-at}\int_{-\infty}^{t} F_s(u)e^{au}\,du + F_s(t) =$$

$$= -ae^{-at}A - e^{-at}\left[F_s(u)e^{au}\Big|_1^t - \int_1^t F_s'(u)e^{au}\,du\right] + F_s(t) =$$

$$= -ae^{-at}A + e^{a(1-t)}F_s(1) + e^{-at}\int_1^t F_s'(u)e^{au}\,du,$$

where

$$A = \int_{-\infty}^{1} F_s(u)e^{au}\,du.$$

From this last result, it is evident that the inequality

$$\int_0^\infty t\,|\,\Phi'(t)\,|\,dt < \infty$$

will have been verified if it is shown that it holds for the matrix

$$e^{-at}\int_1^t F_s'(u)e^{au}\,du.$$

But since

$$\int_0^\infty t\,|\,F_s'(t)\,|\,dt < \infty,$$

we have

$$\int_1^\infty te^{-at}\,dt\int_1^t |\,F_s'(u)\,|\,e^{au}\,du = \int_1^\infty |\,F_s'(u)\,|\,e^{au}\,du\int_u^\infty te^{-at}\,dt =$$

$$= \int_1^\infty |\,F_s'(u)\,|\,e^{au}\left(\frac{1}{a}ue^{-au} + \frac{1}{a^2}e^{-au}\right)du < \infty.$$

Similarly,

$$\int_0^1 te^{-at}dt\int_1^t |\,F_s'(u)\,|\,e^{au}\,du < \infty.$$

q.e.d.

COROLLARY: *If $S(\lambda)$ is a unitary matrix possessing the properties I_s and II_s, then so does*

$$\mathbf{S}(\lambda) = S(\lambda)\left(\frac{\lambda+i}{\lambda-i}\right)^{2m} \tag{2}$$

for any positive integer m.

LEMMA 2: *Let $S(\lambda)$ be a unitary matrix which (a) is continuous over the entire real axis, (b) approaches I as $\lambda \to \pm\infty$, and (c) has the property I_s. Let $F(t)$ be the matrix having $I-S(\lambda)$ as its Fourier transform.*

If

$$-x(t)+\int_0^\infty x(\xi)F_s(-t-\xi)\,d\xi = 0, \qquad 0 \leq t < \infty \tag{3}$$

has q linearly independent solutions in $L^2_{(n)}(0, \infty)$,[1] *then for $m \geq q$, the matrix $\mathbf{S}(\lambda)$ given by (2) possesses the properties I_s and III_s.*

If, in addition, $S(\lambda)$ has the property II_s, then so will $\mathbf{S}(\lambda)$.

Proof: According to the corollary to Lemma 1, it is merely necessary to show that when $m \geq q$, $\mathbf{S}(\lambda)$ has the property III_s. Let $\mathbf{F}_s(t)$ denote the matrix having $I-\mathbf{S}(\lambda)$ as its Fourier transform, and consider the equation

$$-y(t)+\int_0^\infty y(\xi)\mathbf{F}_s(-t-\xi)\,d\xi = 0, \qquad 0 \leq t < \infty.$$

We must prove that it has no non-trivial solution in $L^2_{(n)}(0, \infty)$.

Suppose that $y(t)$ is a solution of this equation belonging to $L^2_{(n)}(0, \infty)$. Then according to Lemma 3.4.1, we have[2]

$$\tilde{y}(\lambda)+\tilde{y}(-\lambda)\mathbf{S}(-\lambda) = 0,$$

where

$$\tilde{y}(\lambda) = \int_0^\infty y(t)e^{-i\lambda t}\,dt.$$

---

[1] From the conditions imposed on $S(\lambda)$ it follows that $F_S(-t)$ satisfies all of the requirements of Lemma 3.3.3 and hence the operator

$$\int_0^\infty x(\xi)F_S(-t-\xi)\,d\xi$$

is completely continuous in $L^2_{(n)}(0, \infty)$.

[2] See footnote 17 of Chapter III.

Using (2), we find from this that the vector

$$\tilde{x}(\lambda) = \tilde{y}(\lambda)\left(\frac{\lambda+i}{\lambda-i}\right)^m$$

satisfies the equation

$$\tilde{x}(\lambda)+\tilde{x}(-\lambda)S(-\lambda) = 0.$$

It is also evident that $\tilde{x}(\lambda)$ is regular along with $\tilde{y}(\lambda)$ in the lower halfplane and is square summable over the real axis. Since $\tilde{x}(\lambda)$ has a zero of m-th order at $\lambda = -i$, the vector functions

$$\tilde{x}^{(k)}(\lambda) = (\lambda^2+1)^{-k}\tilde{x}(\lambda) \qquad (k = 0, 1, \dots, m)$$

are also regular in the lower halfplane, are square summable on the real axis and clearly satisfy

$$\tilde{x}^{(k)}(\lambda)+\tilde{x}^{(k)}(-\lambda)S(-\lambda) = 0 \qquad (k = 0, 1, \dots, m).$$

Therefore, the functions

$$x^{(k)}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{x}^{(k)}(\lambda)e^{i\lambda t}\,d\lambda \qquad (k=0, 1, \dots, m)$$

vanish for $t < 0$, belong to $L^2_{(n)}(0, \infty)$ and according to Lemma 3.4.1, satisfy equation (3).

It merely remains to observe now that if $\tilde{x}(\lambda) \not\equiv 0$, then $\tilde{x}^{(0)}(\lambda)$, $\tilde{x}^{(1)}(\lambda), \dots, \tilde{x}^{(m)}(\lambda)$ would be linearly independent and, hence, so would $x^{(0)}(t), x^{(1)}(t), \dots, x^{(m)}(t)$. Therefore the equation (3) would have at least $m+1$ linearly independent solutions in $L^2_{(n)}(0, \infty)$ and this would contradict the assumption of the lemma. Thus, $\tilde{x}(\lambda) \equiv 0$. This implies that $\tilde{y}(\lambda) \equiv 0$ and so $y(t) \equiv 0$, q.e.d.

COROLLARY: *If $S(\lambda)$ satisfies the same conditions as in Lemma 2, then it can be represented in the form*

$$S(\lambda) = E^{-1}(-\lambda)E(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{2m}. \tag{4}$$

Here, m is a sufficiently large positive integer, and $E(\lambda) = E(0, \lambda)$, where $E(x, \lambda)$ is the special solution of a certain equation of the form (2.1.3) *with a Hermitian potential satisfying* (2.1.2).

In fact, under the stated conditions, $S(\lambda)$ (of formula (2)) has the properties I_s, II_s, III_s, for m a sufficiently large integer and according to Theorem 5.6.1, it is the scattering matrix for some non-singular boundary-value problem.

Therefore by (2.3.3),

$$\mathrm{S}(\lambda) = E^{-1}(-\lambda)E(\lambda),$$

where $E(\lambda)$ has the above meaning. From this and (2) we obtain the relation (4).

§ 2. Indices of $S(\lambda)$

We shall henceforth assume the unitary matrix $S(\lambda)$ $(-\infty<\lambda<\infty)$ to satisfy all of the assumptions of Lemma 2, and we shall look for those solutions of

$$\tilde{z}(\lambda) = \tilde{z}(-\lambda)S(\lambda), \qquad -\infty < \lambda < \infty \tag{5}$$

which are regular in the lower halfplane and are of *finite order at infinity*. This last statement means that the ratio of $\tilde{z}(\lambda)$ to $|\lambda|^{\nu}$, where ν is some constant, tends to zero as $|\lambda|\to\infty$, Im $\lambda\leq 0$. For short, such solutions will be called *proper*.

Lemma 3. *Every proper solution of* (5) *is of the form*

$$\tilde{z}(\lambda) = (\lambda-i)^{2m}\left[\sum_{k=1}^{l}\frac{2\lambda_k}{\lambda^2-\lambda_k^2}\,c^{(k)}N_k + p(\lambda^2)\right]E(\lambda), \tag{6}$$

where $E(\lambda)$ *has the same meaning as in formula* (4), $\lambda_k=-i\mu_k$, $\mu_k>0$ $(k=1, 2, \ldots, l)$, *are the poles of* $E^{-1}(\lambda)$, N_k *is the residue of* $E^{-1}(\lambda)$ *at the pole* λ_k, $p(\lambda)$ *is a vector with components that are polynomials in* λ, *and*

$$c^{(k)} = (\lambda-i)^{-2m}\tilde{z}(\lambda)\,\big|_{\lambda=\lambda_k}.$$

Proof: The relations (4) and (5) yield

$$(\lambda-i)^{-2m}\tilde{z}(\lambda)E^{-1}(\lambda) = (\lambda+i)^{-2m}\tilde{z}(-\lambda)E^{-1}(-\lambda), \qquad \mathrm{Im}\,\lambda = 0.$$

Hence, we see that $\lambda\varphi(\lambda)$, with

$$\varphi(\lambda) = \begin{cases} (\lambda-i)^{-2m}\tilde{z}(\lambda)E^{-1}(\lambda), & \mathrm{Im}\,\lambda\leq 0\\ (\lambda+i)^{-2m}\tilde{z}(-\lambda)E^{-1}(-\lambda), & \mathrm{Im}\,\lambda > 0\end{cases}$$

is an odd function on the real axis. From the properties of $E^{-1}(\lambda)$ (see §§2 and 4 of Chapter II) it then follows that $\lambda\varphi(\lambda)$ is meromorphic in the entire plane, with a finite number of simple poles at the points $\pm\lambda_k$ $(k=1, 2, \ldots, l)$. Therefore

$$\lambda\varphi(\lambda) = \lambda\sum_{k=1}^{l}\frac{2\lambda_k}{\lambda^2-\lambda_k^2}\,c^{(k)}N_k+\psi(\lambda),$$

where N_k is the residue of $E^{-1}(\lambda)$ at the pole λ_k, $\psi(\lambda)$ is an entire odd function, and

$$c^{(k)} = (\lambda - i)^{-2m}\tilde{z}(\lambda)\,|_{\lambda=\lambda_k}.$$

We further note that since $E(\lambda)\to I$ as $|\lambda|\to\infty$ (Im $\lambda\leq 0$), $\varphi(\lambda)$ is of finite order at infinity and hence so is $\psi(\lambda)$. But then $\lambda^{-1}\psi(\lambda)= =p(\lambda^2)$, where $p(\lambda)$ is a vector whose components are polynomials, and consequently

$$\varphi(\lambda) = \sum_{k=1}^{l} \frac{2\lambda_k}{\lambda^2-\lambda_k^2}\, c^{(k)}N_k + p(\lambda^2),$$

which is equivalent to (6).

Remark. The formula (6) shows that when $|\lambda|\to\infty$ (Im $\lambda\leq 0$), each of the components $\tilde{z}_j(\lambda)$ of a proper solution $\tilde{z}(\lambda)$ of (5) has the asymptotic behavior

$$\tilde{z}_j(\lambda)\sim\alpha_j\lambda^{2\nu_j},$$

where α_j is a constant and ν_j an integer. The exponent $2\nu_j$ in this asymptotic relation is said to be the *order* of $\tilde{z}_j(\lambda)$ at infinity, and the largest order of the components of the vector $\tilde{z}(\lambda)$ is said to be the *order* of the vector at infinity. It is evident that the order of any proper solution of equation (5) at infinity is an even integer.

From (4), which is expressible as

$$(\lambda - i)^{2m}E(\lambda) = (\lambda + i)^{2m}E(-\lambda)S(\lambda),$$

it follows that the rows of the matrix $(\lambda - i)^{2m}E(\lambda)$ are proper solutions of (5). These vectors, which will be denoted by

$$e^{(1)}(\lambda),\ e^{(2)}(\lambda), \ldots, e^{(n)}(\lambda), \tag{7}$$

are linearly independent, are regular in the lower halfplane and have exactly the same order $2m$ at infinity. The vectors of (7) also cannot be connected by any relation of the form

$$\sum_{j=1}^{n} p_j(\lambda)e^{(j)}(\lambda)\equiv 0,$$

where $p_j(\lambda)$ are polynomials that are not all simultaneously identically zero. This is a direct consequence of the asymptotic behavior of these vectors at infinity.

We shall now essentially apply the reasoning used in the solution of the Hilbert problem by the method of Plemelj following the presentation in Muskhelishvili's book, *Singular Integral Equations* (1953, § 127, pp 393—404).

From (6), one concludes that among the proper solutions of (5) there will exist those that have the smallest order of the possible orders at infinity. Let $-2\varkappa_1$ denote the smallest order, and let $\tilde{z}^{(1)}(\lambda)$ be a proper solution of (5) having this order at infinity.

Next, let $-2\varkappa_2$ be the smallest possible order at infinity of those proper solutions of (5) which cannot be obtained from $\tilde{z}^{(1)}(\lambda)$ by multiplying by a polynomial, and let $\tilde{z}^{(2)}(\lambda)$ be one of the proper solutions of (5) of the order $-2\varkappa_2$. Evidently, $\varkappa_2 \leq \varkappa_1$. Continuing, let $-2\varkappa_3$ denote the smallest possible order at infinity of those solutions of (5) which cannot be represented in the form

$$p_1(\lambda)\tilde{z}^{(1)}(\lambda)+p_2(\lambda)\tilde{z}^{(2)}(\lambda),$$

where $p_1(\lambda)$ and $p_2(\lambda)$ are polynomials, and let $\tilde{z}^{(3)}(\lambda)$ be one of the proper solutions of order $-2\varkappa_3$.

This process may be continued until we obtain n solutions of (5),

$$\tilde{z}^{(1)}(\lambda),\ \tilde{z}^{(2)}(\lambda), \ldots,\ \tilde{z}^{(n)}(\lambda), \tag{8}$$

having the respective orders at infinity of $-2\varkappa_1$, $-2\varkappa_2$, ..., $-2\varkappa_n$, with

$$\varkappa_1 \geq \varkappa_2 \geq \ldots \geq \varkappa_n .$$

In fact, suppose that the solutions $\tilde{z}^{(1)}(\lambda)$, $z^{(2)}(\lambda), \ldots, \tilde{z}^{(s)}(\lambda)$ have been found in the indicated way and that $s<n$. Then there must exist vectors among those of (7) which can not be represented as

$$\sum_{j=1}^{s} p_j(\lambda)\tilde{z}^{(j)}(\lambda),$$

where the $p_j(\lambda)$ are polynomials. In the contrary case, we would have

$$\sum_{j=1}^{n} q_j(\lambda)e^{(j)}(\lambda) \equiv 0.$$

where the $q_j(\lambda)$ are polynomials which are all not simultaneously zero and this is impossible.

According to Lemma 3, each of the solutions in (8) is of the form

$$\tilde{z}^{(j)}(\lambda) = r^{(j)}(\lambda)E(\lambda) \qquad (j=1, 2, \ldots, n), \tag{9}$$

where $r^{(j)}(\lambda)$ is a vector whose components are rational functions of λ. Let us further set

$$w^{(j)}(\lambda)=\lambda^{2\varkappa_j}\mathfrak{z}^{(j)}(\lambda) \qquad (j=1, 2, \ldots, n). \tag{10}$$

By the use of the properties of the functions in (8), which are immediate consequences of the way in which they were constructed, and by means of (6), (9), (10), and equation (5) itself, it is possible to prove the following assertions:

PROPERTY 1°: *The determinant*

$$\Delta(\lambda)=\det\|\mathfrak{z}_i^{(j)}(\lambda)\|_1^n,$$

where $\mathfrak{z}^{(j)}(\lambda)=\{\mathfrak{z}_1^{(j)}(\lambda), \mathfrak{z}_2^{(j)}(\lambda), \ldots, \mathfrak{z}_n^{(j)}(\lambda)\}$, *does not vanish for all finite values of* $\lambda\neq 0$ *and* $\operatorname{Im}\lambda\leq 0$.

PROPERTY 2°: *The determinant*

$$\Delta_0(\lambda)=\det\|r_i^{(j)}(\lambda)\|_1^n,$$

where $r^{(j)}(\lambda)=\{r_1^{(j)}(\lambda), r_2^{(j)}(\lambda), \ldots, r_n^{(j)}(\lambda)\}$, *does not vanish for all values of* λ *in the halfplane* $\operatorname{Im}\lambda\leq 0$.

PROPERTY 3°: *The determinant*

$$\Delta_\infty(\lambda)=\det\|w_i^{(j)}(\lambda)\|_1^n,$$

where the limit of $w^{(j)}(\lambda)=\{w_1^{(j)}(\lambda), w_2^{(j)}(\lambda), \ldots, w_n^{(j)}(\lambda)\}$ *exists as* $\lambda\to\infty$, $\operatorname{Im}\lambda\leq 0$, *does not vanish.*

The proofs of these statements will be omitted since they essentially repeat the reasoning carried out in the above-mentioned book of Muskhelishvili.

From Property 3° follows the almost evident

COROLLARY: *The order of*

$$\sum_{j=1}^{n} p_j(\lambda)\mathfrak{z}^{(j)}(\lambda)$$

at infinity, where the $p_j(\lambda)$ *are polynomials, is equal to the highest of the orders at infinity of the terms in the sum.*

Any n vector solutions of equation (5) will be called a *canonical system of solutions* of that equation if it has the Properties 1°, 2°, and 3°, and a matrix whose rows comprise a canonical system will be called a *canonical matrix* of equation (5). Now the vector functions in (8), in particular, comprise a canonical system of solutions of (5), and

$$Z(\lambda)=\|\mathfrak{z}_i^{(j)}(\lambda)\|_1^n$$

is a canonical matrix of this equation.

The matrix $Z(\lambda)$ is regular in the lower halfplane and continuous there up to and including the real axis (see formula (6)). From the Properties 1° and 2° it follows that its inverse $Z^{-1}(\lambda)$ is also regular in the lower halfplane and continuous up to and including the real axis, with the possible exception of the point $\lambda = 0$ in the neighborhood of which it behaves like $E^{-1}(\lambda)$. Since the rows of $Z(\lambda)$ are solutions of (5),

$$Z(\lambda) = Z(-\lambda)S(\lambda), \qquad -\infty < \lambda < \infty \tag{11}$$

and hence, for any real finite value of $\lambda \neq 0$

$$S(\lambda) = Z^{-1}(-\lambda)Z(\lambda).$$

By the use of this relation and the relation

$$Z(\lambda) = R(\lambda)E(\lambda), \tag{12}$$

where $R(\lambda) = ||r_i^{(j)}(\lambda)||_1^n$ and which is equivalent to (9), it is now possible to prove the following lemma.

LEMMA 4: *All of the proper solutions of equation (5) can be represented in the form*

$$\mathfrak{z}(\lambda) = \sum_{j=1}^{n} p_j(\lambda^2)\mathfrak{z}^{(j)}(\lambda),$$

where the $p_j(\lambda)$ are polynomials and $\mathfrak{z}^{(j)}(\lambda)$ the canonical system of solutions (8).

The proof of this lemma is carried out in exactly the same way as that of the analogous proposition in Muskhelishvili's book, and we shall omit it.

REMARK. By virtue of this lemma and the corollary to Property 3° formulated above, any proper solution $\mathfrak{z}(\lambda)$ of (5) whose order at infinity does not exceed a given even number 2ν, is of the form

$$\mathfrak{z}(\lambda) = \sum_{j=1}^{n} p_{\nu+\varkappa_j}(\lambda^2)\mathfrak{z}^{(j)}(\lambda),$$

where $p_{\nu+\varkappa_j}(\lambda)$ is a polynomial of degree no higher than $\nu + \varkappa_j$.

By making use of this remark, we can compute the number of linearly independent proper solutions of equation (5) vanishing at infinity.

Let

$$\varkappa_1 \geq \varkappa_2 \geq \dots \geq \varkappa_h \geq 0 > \varkappa_{h+1} \geq \dots \geq \varkappa_n . \tag{13}$$

Any proper solution of (5) vanishing at infinity (i.e., of order not exceeding $2\nu = -2$) is given by

$$\widetilde{z}(\lambda) = \sum_{j=1}^{k} p_{\varkappa_j-1}(\lambda^2)\widetilde{z}^{(j)}(\lambda),$$

where $p_{\varkappa_j-1}(\lambda)$ is a polynomial of degree not exceeding $\varkappa_j - 1$. Moreover, $p_{\varkappa_j-1}(\lambda) \equiv 0$ if $\varkappa_j - 1 < 0$. This expression contains

$$N_{(+)} = \varkappa_1 + \varkappa_2 + \ldots + \varkappa_k \tag{14}$$

arbitrary constants [the coefficients of the polynomials $p_{\varkappa_j-1}(\lambda)$] and one can easily see that this is the number of linearly independent proper solutions of (5) vanishing at infinity.

Let us now turn again to equation (11). Taking first the complex conjugate transpose and then the inverse of both its sides and finally multiplying by λ, we arrive at

$$\lambda[Z^*(\lambda)]^{-1} = \lambda[Z^*(-\lambda)]^{-1}S(\lambda) \qquad (\operatorname{Im}\lambda = 0,\ \lambda \neq 0)$$

(since $S^*(\lambda) = S^{-1}(\lambda)$), or

$$\lambda[Z^*(-\overline{\lambda})]^{-1} = \lambda[Z^*(\overline{\lambda})]^{-1}S(-\lambda) \qquad (\operatorname{Im}\lambda = 0,\ \lambda \neq 0).$$

This implies that the rows of the matrix

$$Y(\lambda) = \lambda[Z^*(-\overline{\lambda})]^{-1} \tag{15}$$

are solutions (and furthermore, proper ones) of the equation

$$\widetilde{y}(\lambda) = -\widetilde{y}(-\lambda)S(-\lambda). \tag{16}$$

It immediately follows from (15) that for any finite non-zero value of λ in the halfplane $\operatorname{Im}\lambda \leq 0$,

$$\det Y(\lambda) \neq 0.$$

Now set

$$Y(\lambda) = \| \widetilde{y}_i^{(j)}(\lambda) \|_1^n ;$$

then (15) yields

$$\widetilde{y}_i^{(j)}(\lambda) = \frac{\lambda\overline{\Delta}_{ij}(-\overline{\lambda})}{\overline{\Delta}(-\overline{\lambda})},$$

where $\Delta_{ij}(\lambda)$ is the cofactor of the element $\widetilde{z}_i^{(j)}(\lambda)$ in the determinant $\Delta(\lambda) = \det Z(\lambda)$. But by virtue of (10),

$$\widetilde{y}_i^{(j)}(\lambda) = \frac{\lambda\overline{\Delta}_{ij}^{\infty}(-\overline{\lambda})}{\overline{\Delta}_{\infty}(-\overline{\lambda})}\,\lambda^{2\varkappa_j},$$

where $\Delta_{ij}^{\infty}(\lambda)$ is the cofactor of the element $w_i^{(j)}(\lambda)$ in the determinant $\Delta_\infty(\lambda)$. Hence, it follows that the solution $\tilde{y}^{(j)}(\lambda)=\{\tilde{y}_1^{(j)}(\lambda), \tilde{y}_2^{(j)}(\lambda), \dots, \tilde{y}_n^{(j)}(\lambda)\}$ of equation (16) is precisely of the order $2\varkappa_j+1$ at infinity and that

$$\det \| \lambda^{-(2\varkappa_j+1)}\tilde{y}_i^{(j)}(\lambda) \|_1^n = \frac{1}{\bar{\Delta}_\infty(-\bar{\lambda})}$$

does not vanish there.

Thus the matrix $Y(\lambda)$ defined by (15) possesses the Properties 1° and 3° of a canonical matrix. Also holding is the following lemma.

LEMMA 5: *All proper solutions of equation* (16) *can be represented as*

$$\tilde{y}(\lambda)=\sum_{j=1}^{n} p_j(\lambda^2)\tilde{y}^{(j)}(\lambda),$$

where the $p_j(\lambda)$ *are polynomials and the* $\tilde{y}^{(j)}(\lambda)$ *rows of the matrix* $Y(\lambda)$.

By means of these results, it is possible to show as in the above that any proper solution of (16) vanishing at infinity can be represented in the form (see (13))

$$\tilde{y}(\lambda)=\sum_{j=k+1}^{n} p_{|\varkappa_j|-1}(\lambda^2)\tilde{y}^{(j)}(\lambda),$$

where $p_{|\varkappa_j|-1}(\lambda)$ is an arbitrary polynomial of degree not exceeding $|\varkappa_j|-1$. The total number of arbitrary coefficients is therefore

$$N_{(-)} = -\varkappa_{k+1}-\varkappa_{k+2}-\dots-\varkappa_n, \tag{17}$$

and this is the number of linearly independent proper solutions of (16) vanishing at infinity.

From (14) and (17), we thus have

$$N_{(+)}-N_{(-)} = \varkappa_1+\varkappa_2+\dots+\varkappa_n .$$

The numbers $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ are called the *partial indices*, and their sum the *total index* of the matrix $S(\lambda)$.

According to Lemma 3.4.1, $N_{(+)}$ is the number of linearly independent solutions $z(t)$ of

$$z(t)+\int_0^\infty z(\xi)F_s(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty, \tag{18}$$

belonging to $L^2_{(n)}(0, \infty)$, and $N_{(-)}$ the number of linearly independent solutions $y(t)$ of

$$-y(t)+\int_0^\infty y(\xi)F_s(-t-\xi)\,d\xi = 0, \qquad 0 \leq t < \infty,$$

belonging to $L^2_{(n)}(0, \infty)$, or what is the same, the number of linearly independent solutions $\hat{y}(t)$ of

$$-\hat{y}(t)+\int_{-\infty}^0 \hat{y}(\xi)F_s(t+\xi)\,d\xi = 0, \qquad -\infty < t \leq 0, \tag{19}$$

belonging to $L^2_{(n)}(-\infty, 0)$, where $F_s(t)$ is the matrix having $I-S(\lambda)$ as its Fourier transform.

Thus we have proved

THEOREM 1: *Suppose that the unitary matrix $S(\lambda)$ satisfies all of the assumptions of Lemma* 2 *and $F_s(t)$ has $I - S(\lambda)$ as its Fourier transform. Then*

$$N_{(+)}-N_{(-)} = \varkappa,$$

where $N_{(+)}$ is the number of linearly independent solutions of (18) *in $L^2_{(n)}(0, \infty)$, $N_{(-)}$ the number of linearly independent solutions of* (19) *in $L^2_{(n)}(-\infty, 0)$, and $\varkappa$ is the total index of $S(\lambda)$.*

If $S(\lambda)$ satisfies the conditions of Lemma 2, then from our earlier discussion

$$S(\lambda) = E^{-1}(-\lambda)E(\lambda)\left(\frac{\lambda-i}{\lambda+i}\right)^{2m}, \tag{4}$$

where $E(\lambda) = E(0, \lambda)$ and $E(x, \lambda)$ is the special solution of some non-singular equation of the form (2.1.3). The computation of the total index of the matrix $S(\lambda)$ requires the following lemma.

LEMMA 6: *In the neighborhood of $\lambda = 0$ with* Im $\lambda \leq 0$,

$$\det E(\lambda) = \lambda^q\{c_0+o(1)\}, \qquad c_0 \neq 0,$$

where q is the rank of $I - S(0)$.

Proof: Let P_0 be a matrix projecting from the left on the null space of $E(0)$. According to Lemmas 5.6.2 and 5.6.3, the matrix

$$D(\lambda) = [I-P_0+(i\lambda)^{-1}P_0]E(\lambda)$$

is continuous and non-singular in the neighborhood of $\lambda = 0$. Therefore,

$$\det E(\lambda) = \det[I-P_0+(i\lambda)^{-1}P_0]^{-1}\det D(\lambda),$$

But the $\lim\limits_{\lambda \to 0} \det D(\lambda) = c_0' \neq 0$ and hence

$$\det E(\lambda) = \det[I - P_0 + i\lambda P_0] \cdot (c_0' + o(1)) \qquad (\lambda \to 0).$$

Let q be the rank of the matrix P_0. Then in an appropriately chosen basis, P_0 will be in diagonal form with q of its elements equal to 1 and the remaining ones 0. Since the value of the determinant of a matrix remains unchanged under a unitary transformation of basis, it follows that

$$\det[I - P_0 + i\lambda P_0] = (i\lambda)^q.$$

Therefore when $\lambda \to 0$

$$\det E(\lambda) = \lambda^q(c_0 + o(1)),$$

where $c_0 = i^q c_0' \neq 0$. Furthermore, (5.6.20) yields

$$q = \operatorname{rank} P_0 = \operatorname{rank}[I - \lim_{\lambda \to 0} E^{-1}(-\lambda)E(\lambda)] = \operatorname{rank}[I - S(0)],$$

and the proof of the lemma is complete.

We now make use of equation (11) to compute the total index of $S(\lambda)$. According to (11),

$$\log \det Z(\lambda) = \log \det Z(-\lambda) + \log \det S(\lambda). \tag{20}$$

Noting that $|\det S(\lambda)| = 1$ and setting

$$\eta(\lambda) = \arg \det S(\lambda) = -i \log \det S(\lambda),$$

we find that for any given $R > \varepsilon > 0$

$$\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} d(\log \det S(\lambda)) = i[\eta(R) - \eta(\varepsilon) + \eta(-\varepsilon) - \eta(-R)]. \tag{21}$$

By means of the properties of the canonical matrix $Z(\lambda)$, it is possible to set up a similar integral for $\log \det Z(\lambda)$ using contour integration. Namely, if l_R and l_ε are semicircles in the lower half-plane with centers at the origin and with respective radii of R and ε and if l_R is traversed clockwise and l_ε counterclockwise, then

$$\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} d(\log \det Z(\lambda)) = -\int_{l_R} - \int_{l_\varepsilon} d(\log \det Z(\lambda)).$$

By the Property 3° of the canonical matrix $Z(\lambda)$,

$$\det Z(\lambda) \sim \gamma \lambda^{-2\varkappa}$$

when $\lambda \to \infty$, Im $\lambda \leq 0$, where γ is a constant, and hence

$$\lim_{R\to\infty} \int_{l_R} d(\log \det Z(\lambda)) = 2\pi i\varkappa.$$

Now by equation (12),

$$\det Z(\lambda) = \log \det R(\lambda) + \log \det E(\lambda),$$

and hence, by applying Lemma 6 and the fact that det $R(0) \neq 0$, we find that

$$\lim_{\varepsilon\to 0} \int_{l_\varepsilon} d(\log \det Z(\lambda)) = \pi iq.$$

Thus

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left\{ \int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} d(\log \det Z(\lambda)) \right\} = -2\pi i\varkappa - \pi iq. \tag{22}$$

Similarly, by means of contour integration in the upper half-plane, it may be shown that

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left\{ \int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} d(\log \det Z(-\lambda)) \right\} = 2\pi i\varkappa + \pi iq. \tag{23}$$

From equations (20)—(23), there then results

$$\eta(+\infty) - \eta(+0) + \eta(-0) - \eta(-\infty) = -4\pi\left(\varkappa + \frac{1}{2}q\right). \tag{24}$$

Since $S(\lambda)S(-\lambda) = I$ for all real values of λ,

$$\eta(\lambda) + \eta(-\lambda) = 2\pi m, \tag{25}$$

where m is an integer. We may also assume $\eta(\lambda)$ to be a continuous function since det $S(\lambda)$ is continuous and does not vanish.

Therefore, if the relation (25) for $\lambda = \infty$ is combined with (24), and then the relation (25) for $\lambda = 0$ is subtracted from the resulting expression, we obtain

$$\eta(0) - \eta(+\infty) = 2\pi\left(\varkappa + \frac{1}{2}q\right),$$

or

$$\varkappa = \frac{1}{2\pi}[\eta(0) - \eta(+\infty)] - \frac{1}{2}q. \tag{26}$$

§ 3. A New Characterization of the Scattering Data

The above results lead to a modification of the set of characteristic properties of the scattering data of the non-singular boundary-value problem.

According to Theorem 5.6.2, the scattering matrix $S(\lambda)$ of the non-singular problem is continuous over the entire real axis and approaches I when $\lambda \to \pm\infty$. Thus, Theorem 1 of this appendix is applicable to it. From this theorem and the properties III_s and V of the scattering data of the non-singular problem, it follows that the sum of the ranks of the normalization matrices M_1, M_2, $\ldots$, M_p of this problem is equal to the total index $\varkappa$ of $S(\lambda)$ which can be computed by means of formula (26).

Let there now be given Hermitian matrices M_k, numbers $\lambda_k^2 < 0$ ($k = 1, 2, \ldots, p$) and a unitary matrix $S(\lambda)$ continuous over the entire real axis, such that $S(\lambda) \to I$ when $\lambda \to \pm\infty$, and satisfying I_s, II_s, and IV. Suppose further that the sum of the ranks of M_1, M_2, $\ldots$, M_p is equal to the total index $\varkappa$ of $S(\lambda)$. Then, as we next show, the conditions III_s and V will also hold.

According to Theorem 1, the number of linearly independent solutions of equation (18) in $L^2_{(n)}(0, \infty)$, i.e. $N_{(+)}$, is not less than $\varkappa$. However, this number cannot exceed $\varkappa$. In fact, if

$$z^{(1)}(t),\ z^{(2)}(t),\ \ldots,\ z^{(l)}(t)$$

were linearly independent solutions of (18) and $l > \varkappa$, then there would exist a non-trivial linear combination of these solutions

$$z(t) = \sum_{j=1}^{l} \gamma_j z^{(j)}(t),$$

such that

$$\tilde{z}(\lambda_k) M_k = 0 \qquad (\operatorname{Im} \lambda_k < 0;\ k = 1, 2, \ldots, p),$$

where

$$\tilde{z}(\lambda) = \int_0^\infty z(t) e^{-i\lambda t}\, dt.$$

This is so because the rank of the following system of equations in the unknowns γ_j does not exceed $\varkappa$ (i.e., the sum of the ranks of the matrices M_k):

$$\sum_{j=1}^{l} \gamma_j \tilde{z}^{(j)}(\lambda_k) M_k = 0 \qquad (k = 1, 2, \ldots, p).$$

But then according to the corollary to Lemma 3.4.1, the condition IV would be violated.

Thus $N_{(+)} = \varkappa$, and hence the condition V holds since $\varkappa$ is the sum of the ranks of $M_1, M_2, \ldots, M_k$. Furthermore, by Theorem 1, the fact that $N_{(+)} = \varkappa$ implies that $N_{(-)} = 0$, i.e., the condition III_s holds.

These results together with Theorem 5.6.2 yield the following particular theorem.

THEOREM 2: *A unitary matrix* $S(\lambda)$, *Hermitian matrices* M_k, *and numbers* $\lambda_k^2 < 0$ $(k = 1, 2, \ldots, p)$ *comprise the scattering data for some boundary-value problem* (2.1.1) *with a Hermitian potential* $V(x)$ *satisfying the inequality*

$$\int_0^\infty x\,|V(x)|\,dx < \infty,$$

if and only if the following conditions hold:

(*1*) $I - S(\lambda)$ *is the Fourier transform of a Hermitian matrix* $F_s(t)$ *summable over the entire real axis* $(-\infty < t < \infty)$.

(*2*) *For all values of* $t > 0$, $F_s'(t)$ *exists and*

$$\int_0^\infty t\,|F_s'(t)|\,dt < \infty.$$

(*3*) *The equation*

$$x(t) + \int_0^\infty x(\xi)F(t+\xi)\,d\xi = 0, \qquad 0 \le t < \infty,$$

in which

$$F(t) = \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k| t} + F_s(t)$$

has no non-trivial solution in $\hat{L}_{(n)}(0, \infty)$.

(*4*) *The sum of the ranks of the matrices* $M_1, M_2, \ldots, M_p$ *is given by*

$$\frac{1}{2\pi}[\eta(0) - \eta(+\infty)] - \frac{1}{2}q,$$

where $i\eta(\lambda) = \log\det S(\lambda)$ *and* q *is the rank of* $I - S(0)$.

APPENDIX II

Refinement of Certain Inequalities

In the course of our investigation of the non-singular boundary-value problem, the following two inequalities were obtained (see (3.2.7) and (5.4.4′)):

$$|F'(2x)-\frac{1}{4}V(x)| \leq C_1\left\{\int_x^\infty |V(t)|\,dt\right\}^2, \tag{1}$$

$$|V(x)-4F'(2x)| \leq C_2\left\{\int_x^\infty |F'(2t)|\,dt\right\}^2. \tag{2}$$

Here C_1 and C_2 are constants, $V(x)$ is the potential of the problem in question, and $F(x)$ is a matrix which is expressed in terms of the scattering data for this problem by means of the formulas

$$\begin{aligned} F(x) &= \sum_{k=1}^{p} M_k^2 e^{-|\lambda_k|x} + F_s(x), \\ F_s(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [I-S(\lambda)]e^{i\lambda x}\,d\lambda. \end{aligned} \tag{3}$$

The above inequalities show that $V(x)$ and $4F'(2x)$ behave essentially alike in the neighborhood of zero and infinity.

We recall that (1) and (2) are derived from the fundamental equation

$$F(x+y)+K(x,\ y)+\int_x^\infty K(x,\ t)F(t+y)\,dt = 0, \tag{4}$$

the first on the assumption that

$$\int_0^\infty x\,|V(x)|\,dx < \infty,$$

and the second provided that

$$\int_0^\infty x\,|F'(x)|\,dx < \infty.$$

It is of natural interest to seek a refinement of the inequalities (1) and (2), in other words, to obtain asymptotic relations between $V(x)$ and $F'(2x)$ containing the next terms in the asymptotic expansions. We shall now give without proof several of such formulas.

By means of a first iteration of the fundamental equation (4), it can be shown that

$$\begin{aligned} &\left| F'(2x) - \frac{1}{4} V(x) + \frac{1}{4} \left\{ \int_x^\infty V(t)\,dt \right\} \right| \leq \\ &\leq C_3 \left\{ \int_x^\infty |V(t)|\,dt \right\}^2 \int_x^\infty t\,|V(t)|\,dt, \end{aligned} \tag{5}$$

$$\begin{aligned} &|V(x) - 4F'(2x) - 4F^2(2x)| \leq \\ &\leq C_4 \left\{ \int_x^\infty |F'(2t)|\,dt \right\}^2 \int_x^\infty t\,|F'(2t)|\,dt, \end{aligned} \tag{6}$$

which clearly sharpen (1) and (2) for large values of x only.

To obtain more precise formulas for x in the neighborhood of zero, it is necessary to impose a more stringent a priori condition on the one of the two matrices $V(x)$ and $F'(2x)$ whose behavior is considered to be known.

Thus, it may be shown that if

$$\int_0 x^{1-\delta}\,|V(x)|\,dx < \infty \qquad \left(\frac{2}{3} < \delta < 1\right), \tag{7}$$

then

$$F'(2x) = \frac{1}{4} V(x) - \frac{1}{4} \left\{ \int_x^\infty V(t)\,dt \right\}^2 + A + O(x^{-2+3\delta}) \quad (x \to 0), \tag{8}$$

where

$$A = \int_0^\infty \left\{ \int_0^\infty V(s)\,[K(s,\ 2t+s) - K(0,\ 2t)]\,ds \right\} F(2t)\,dt \tag{9}$$

and $K(x,\ y)$ is the solution of equation (4).

On the other hand, assuming that $F(x)$ satisfies the condition

$$\int_0 x^{1-\delta}\,|\,F'(x)\,|\,dx < \infty \qquad \left(\frac{2}{3} < \delta < 1\right), \tag{10}$$

one can obtain for $V(x)$ the asymptotic representation

$$\begin{aligned} V(x) &= 4F'(2x) + 4[F(2x) - F(\pm 0)]^2 - \\ &\quad -4A + O(x^{-2+3\delta}) \qquad (x \to 0), \end{aligned} \tag{11}$$

where A is given by (9) and

$$F(\pm 0) = \sum_{k=1}^{p} M_k^2 + \frac{1}{2} \lim_{x \to 0} [F_s(x) + F_s(-x)]$$

(the limit exists whenever the condition (10) holds).

The representation (11) can also be expressed as

$$\begin{aligned} V(x) &= 4F'(2x) + [F_s(2x) - F_s(-2x)]^2 - \\ &\quad -4A + O(x^{-2+3\delta}) \qquad (x \to 0). \end{aligned}$$

BIBLIOGRAPHY

(Items in the Russian language are indicated by △)

[1]. Newton, R.G., Jost, R.: The construction of potentials from the S-matrix for systems of differential equations, *Il Nuovo Cimento*, vol. 1, no. 4 (1955), 590—622.

△[2]. Gelfand, I.M., Levitian, B.M.: On the determination of a differential equation from its spectral function, *Izvestiya AN SSSR, ser. matem.*, vol. 15, no. 4 (1951), 309—360.

△[3]. Krein, M.G.: On the determination of the potential of particles from its S-function, *Doklady AN SSSR*, vol. 105, no. 3 (1955), 433—436.

△[4]. Krein, M.G.: On the theory of accelerants and S-matrices for canonical differential systems, *Doklady AN SSSR*, vol. 111, no. 6 (1956), 1167—1170.

△[5]. Marchenko, V.A.: The construction of the potential energy from the phases of the scattered waves, *Doklady AN SSSR*, vol. 104, no. 5 (1955), 695—698.

△[6]. Agranovich, Z.S., Marchenko, V.A.: The construction of the potential from the scattering matrix for a system of differential equations, *Doklady AN SSSR*, vol. 113, no. 5(1957), 951—954.

△[7]. Levin, B.Ya.: Fourier and Laplace-type transformations using solutions of a second-order differential equation, *Doklady AN SSSR*, vol. 106, no. 2 (1956), 187—190.

[8]. Blatt, G., Weisskopf, V.: *Theoretical Nuclear Physics*, Wiley and Sons, New York 1958.

[9]. Newton, R.G.: Connection between the S-matrix and the tensor force, *Phys. Rev.*, vol. 100, no. 1 (1955), 412—428.

△[10]. Agranovich, Z.S., Marchenko, V.A.: Construction of the tensor forces from the scattering data, *Doklady AN SSSR*, vol. 118, no. 6(1958), 1055—1058.

△[11]. Glazman, I.M.: On the spectrum of linear differential operators, *Doklady AN SSSR*, vol. 80, no. 2 (1951), 153—156.

[12]. Wiener, N.: *The Fourier Integral and Certain of Its Applications*, Cambridge University Press 1933.

[13]. Crum, M.M.: Associated Sturm-Liouville systems, *Quart. Journ. of Math., Oxford* (2), vol. 6, no. 2 (1955), 121—128.

△[14]. Krein, M.G.: On the continuum analogy of a formula of Christoffel in the theory of orthogonal polynomials, *Doklady AN SSSR*, vol. 113, no. 5 (1957), 970—973.

△[15]. Faddeyev, L.D.: The inverse problem of the quantum theory of scattering, *Uspekhi matem. nauk*, vol. 14, no. 4 (88) (1959), 57—119. An English translation of this article appears in *Journ. of Math. Phys*,, vol. 4, no. 1 (1963), 72—104. (This article contains an extensive bibliography.)

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